

A Partial Differential Equation
Modeling
Forest Growth

Diploma Thesis

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Chapter 1

Introduction

1.1 General remarks on forest models

Forest models are used in a number of areas like forest management, estimating whether the forest provides sufficient protection against avalanches and rockfall, and on a larger spatial scale vegetation models are now being used to estimate the buffer capacity of the biosphere in response to the climate change.

A central question in forest modeling is the level of aggregation used. Improving the realism of the model usually leads to decreased analytical tractability; tracing the fate of every tree in a forest, though in principle possible, does not lead to deeper understanding of the forest's fundamental dynamics.

One class of reasonably realistic manageable models are patch models. They model in a rather explicit way the development of a "patch" in the forest. The size of such a patch is taken to be about $\frac{1}{12}$ ha: on the one side, big enough to assume no interaction (other than seed transport) with other patches, on the other side small enough so that every tree inside can be assumed to influence all the others. Thus the horizontal dimension can be effectively eliminated. The fate of every tree is explicitly modeled from birth to death. All birth, growth and death processes are modeled in a probabilistic manner. The simulation is run many times and the abundance of each species then averaged over all the simulations to give an approximation of the forest dynamics. Although some simplifying assumptions have to be made (for instance, that all the leaves of a tree are concentrated at one height, so that we can characterize the tree by its height), the patch models are rather realistic, and

the biological meaning of all parameters used is clear. However, due to their probabilistic nature, the patch models are too complicated to allow deep analytic understanding in the near future.

At the next approximation level are the structured population models; they represent the population as a population density y , a function of one or several structure parameters such as age, size etc., together with a partial differential equation (PDE) governing the evolution of y in time. In case of forests the natural structuring parameter is tree height.

The coefficients of the PDE that governs the evolution of a structured model can be obtained by fitting some plausible kind of dependence to the data [1], [2]. A more ambitious method is trying to derive them from a patch model [3].

In either case, it would be desirable to find simple correspondences between some elementary mathematical properties of the coefficient functions used in a height-structured model and the dynamic behavior of the system, to the point of describing exactly what conditions on the coefficient functions lead to a certain behavior. Upon achieving such understanding it would become possible to check assumptions about individual dynamics by trying to translate them into height-structured models and comparing the resulting model's dynamics with observations.

Understanding the behaviour of height-structured models as well as their relation to patch models would also contribute significantly to bridging the gap between the small-scale patch-models of forests and the large-scale description obtained by fitting simple functions to the data, since in height-structured models the level of aggregation is high on one side, and the connection to individual dynamics can still be traced on the other.

The goal of this thesis is providing a sound foundation and some starting points for further research in this direction.

1.2 A general model for height structure dynamics

A general equation for the dynamics of a height-structured model can be found in [4]. It is analytically hardly tractable, but useful as a starting point for construction of simpler models. It is

$$\partial_t y(h, t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \partial_h^n \left(M_n^G(h, t) y(h, t) \right) - M^\mu(h, t) y(h, t) \quad (1.1)$$

where $M^\mu(h, t)$ is the expected value at the time t of mortality of an individual having height h , $M_1^G(h, t) =: G(h, t)$ is the expected value at the time t of the growth rate of an individual having height h , and $M_n^G, n \geq 2$ is the n 'th moment of the growth rate centered around the expected value. (For instance, M_2^G is the variance of the growth rate).

The natural boundary conditions are: Firstly, the total inflow (sapling recruitment) at the left boundary ($h = h_0$):

$$\Phi(t) = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \partial_h^{n-1} \left(M_n^G(h, t) y(h, t) \right) \Big|_{h=h_0} \quad (1.2)$$

Φ being the expected amount of new saplings of height h_0 .

Another natural boundary condition is

$$y(h_{max}, t) = 0 \quad \forall t \quad \text{for some } h_{max} \gg h_0, \quad (1.3)$$

which simply means there is a certain upper bound to the tree size.

All research in this direction up to now has concentrated on approximations of (1.1) omitting all terms of order $n \geq 3$; in this case these two boundary conditions together with some initial condition suffice to determine the evolution of the height structure.

It is to my knowledge an open question what kind of boundary conditions higher-order approximations of (1.1) would need.

Equation (1.1) (or some finite-order approximation thereof) gives a description of height structure dynamics if $M_n^G(h, t)$, $M^\mu(h, t)$ are known.

The main problem is that it is, in general, impossible to compute all these expected values knowing only the height structure. Therefore, to construct an autonomous height-structured model some approximation has to be made. The next section discusses different approaches in the literature.

1.3 Literature review

Very little is known about partial differential equations of order higher than two. Therefore one usually omits all terms with $n > 2$ in the sum of (1.1)

The existing results then fall mainly into two broad categories:

1. Theoretical results for simple models, where a deep understanding of a model or whole classes of models is achieved with (often rather impressive) mathematical techniques . Unfortunately, models allowing such analysis tend to be extremely simplified, so that their use for modeling real-world systems is often restricted.
2. Simulations of rather detailed, realistic models. Unfortunately, simulations only allow one to explore model behaviour for a finite number of coefficient functions/initial conditions, so one can never be quite sure whether the effects observed are just artifacts of the (arbitrary) choices one must make when building a numerical model. Besides, the richness of behaviour produced by a dynamic model may still be too great to lead towards real understanding of the system, if this understanding is not helped by analytical results.

In the following we proceed to review the existing research results on structured population models in both these areas.

1.3.1 Theoretical research on structured population models

All mathematical research on structured population models known to me has only considered first-order equations (keeping only the $n = 1$ -term in the sum of (1.1)). Cushing in [5] even considers this general first order equation to be a “unifying model in population dynamics”.

Mathematical study of the dynamics of a population structured with respect to some internal parameter (mostly age) has a long tradition.

[6] studied a linear growth equation for the density $y(a, t)$ of individuals as a function of time and age:

$$\partial_t y(a, t) + \partial_a y(a, t) = -\mu(a)y(a, t) \quad (1.4)$$

subject to an integro-boundary condition known as “the renewal equation”

$$y(0, t) = \int_{a_{min}}^{a_{max}} y(a, t)\beta(a)da \quad (1.5)$$

as well as some initial condition. A nonlinear self-coupling was introduced in the classical papers [7], [8] by allowing μ and β to depend on the total

population $P(t)$ defined as

$$P(t) := \int_0^{\infty} y(a, t) da \quad (1.6)$$

so that their equation takes the form

$$\partial_t y(a, t) + \partial_a y(a, t) = -\mu(a, P(t))y(a, t) \quad (1.7)$$

with the boundary condition

$$y(0, t) = \int_{a_{min}}^{a_{max}} y(a, t) \beta(a, P(t)) da. \quad (1.8)$$

A lot of research has been done on the theory of both the linear ([9], [10], [11]) and the nonlinear models. Several generalisations have been developed, such as the corresponding many-species systems ([12], [13]) and generalisations to an arbitrary number of continuous structure variables besides age ([14]).

A classical reference on the topic is [15]; an overview of the field can be found in [5].

However, most of the research has been concerned with populations structured according to age (which excludes nonlinear growth rates and has the advantage that the characteristics of the equation are known in advance); and among the papers treating size-structured models with nonlinear growth rates caused by interaction among individuals (e.g. [16], [17], [18], [19]) the nonlinearity has almost invariably been modeled by a global real-valued time-dependent variable characterising “environment quality”. One common choice of this variable is the total population defined above or some other continuous functional $\mathcal{Q}(y(\cdot, t))$; another choice is a global non-structured resource R exploited by all individuals, obeying

$$\frac{d}{dt} R = k(R) - \mathcal{D}(y(\cdot, t)), \quad (1.9)$$

\mathcal{D} being again some continuous functional with the meaning of depletion rate of the resource.

Even [20], who explicitly aim at modeling competition for light in a forest, adhere to this scheme.

A characteristic trait of competition for light in a forest, however, is its *hierarchical* nature, since bigger trees reduce the amount of light available to smaller trees, but are in no way influenced by them as far as light is concerned.

In the simulations of [1] substituting dependence on available light, which at any given moment differs according to height ¹ by dependence on the cumulative basal area at the forest floor (which is a height-independent constant like the “environment quality” above) had a significant effect on the resulting size distribution, the latter dependence producing distributions much less in accordance with the observations.

As the goal of this thesis is studying *realistic* forest models, I will only consider hierarchical models, where the values of all coefficient functions at height h are allowed to depend on h and the height structure above h only. The only mathematically oriented paper dealing with this kind of hierarchical nonlinearity is [21], where an age-structured model with age-dominance is considered and an ordinary differential equation for total population size is derived and studied.

However, the method used relies heavily on the assumption that the vital rates of an individual are *not* explicitly dependent on age, but only on time and the amount of individuals older and/or younger than itself, which is highly unrealistic as far as trees are concerned : Even if one substitutes size for age, the growth rate of a tree, for instance, is clearly strongly size-dependent.

None of the mathematical models studied so far are close enough to the class of models I am going to consider to be directly used in its investigation.

1.3.2 Realistic height-structured forest models

Because there now exist a number of approaches and ideas as to how individual-level properties can/should be aggregated into structured population models, but so far no complete understanding, there exists a very large amount of literature on the subject.

Because of this variety, this review presents only a short sketch of the results known to the author, without any claim of completeness.

One line of research has been followed by Kohyama et al. who in a series

¹in [1] available light is approximated by $L = e^{-C \cdot B(h,t)}$, $B(h,t)$ being the cumulative basal area of all trees above height h

of papers built up a second-order height-structured PDE forest model, first for a generic “species” ([1]), then for several competing species ([2]). The only interaction between trees was assumed to be competition for light. Light availability was approximated by $L = e^{-C \cdot B(h,t)}$, $B(h,t)$ being the cumulative basal area of all trees above height h .

For each of the coefficients, several simple but plausible types of dependence on (B, h) have been tested, the parameters fitted to the empirical data for two different forest types, and the resulting system simulated numerically. Then for each coefficient function the dependency type was chosen that led to simulations best corresponding to the biological data.

Following effects have been observed:

- The system always tends to a steady-state solution that is independent of the initial conditions.
- The second-order term that was used had practically no effect on the dynamics of the system. It has therefore been omitted in later models of these authors.
- In the case where sapling recruitment depended on the light availability in a strictly increasing way (resp. on B in a strictly decreasing way), the system, when started at a small initial tree density, experienced an overshoot before stabilising to steady state. If, however, sapling recruitment was set constant, equal to its steady state value, the system approached the same steady state in a monotone way.
- The exact numerical values of sapling recruitment had little influence on the steady state height structure. To be precise, doubling resp. halving the recruitment rates led to a correspondingly large change in $y_{steady}(h_0)$. However, for increasing h the difference to the “normal” steady state height structure diminished rapidly.

In consequent papers [22], [23], the model was further developed beyond the class of models considered in this thesis.

Another line of research is that of [24], who developed a fully spatial individual-based model of forest dynamics and tried approximating it by a size structured model with coefficient functions equal to the corresponding vital rates of individual trees. This approximation performed very poorly, leading to only half the standing crop of the individual-based model and much quicker extermination of non-dominant species.

A more successful attempt to translate an individual-based model into a height-structured one is due to Lischke et al.([3]) Their model DisCForM has, in fact, served as a starting point and a continuous inspiration for this thesis.

Therefore, I discuss DisCForM in detail in the next section.

1.4 Height-structured model DisCForM

The goal of DisCForM is reproducing the dynamics of the patch model ForClim [25] by means of a first-order PDE system for size structures $y_s : [h_0, H_{max}] \times [0, \infty[\rightarrow \mathbb{R}_+$ of several species, $s = 1, \dots, S$. As in the present thesis only one-species models are considered, I will assume $S = 1$ throughout this section and omit the index s .

The PDE considered has the form

$$\partial_t y(h, t) = \partial_h (G \cdot y(h, t)) - M \cdot y(h, t) \quad (1.10)$$

with the left boundary condition

$$G \cdot y|_{h=h_0} = \Phi \quad (1.11)$$

G , M and Φ are, respectively, the effective growth mortality and birth rates; they are all height- and time dependent (the exact dependencies will be specified later). In the following discussion I refer to them as the coefficient functions, as opposed to $G_{ind}(S, h)$, $M_{ind}(S, h)$, and $\Phi_{ind}(S)$ - the growth rate, mortality and birth probabilities of an individual, which I will call the individual's vital rates.

Here $S = S_{h,t}$ (for ‘‘shadow’’) stands for the cumulative leaf area above height h at time t . It is treated as a random variable having the distribution $\rho_S(S, h, t)$. This distribution is dependent on the height structure above h . A more detailed discussion of ρ_S can be found in the section 2.3. Here I only note that if a tree of height h has a total leaf area $\alpha(h)$, then the expected value of $S_{h,t}$ is

$$E[S](h, t) = \int_0^\infty \rho_S(S, h, t) S dS = \int_0^{H_{max}} y(h', t) \alpha(h') dh' \quad (1.12)$$

The coefficient functions are computed as the expected values of the deterministic individual vital rates

$$G(h, t) := \int_0^\infty \rho_S^*(S, h, t) G_{ind}(S, h) dS \quad (1.13)$$

$$M(h, t) := \int_0^\infty \rho_S^*(S, h, t) M_{ind}(S, h) dS \quad (1.14)$$

$$\Phi(t) := \int_0^\infty \rho_S^*(S, h_0, t) \Phi_{ind}(S) dS \quad (1.15)$$

where ρ_S^* is an approximation of ρ_S by a Normal density function with same expected value and variance as ρ_S that is set to zero for $S < 0$ and renormalized to give $\int_0^\infty \rho_S^*(S, h, t) dS = 1$. For a discussion of the quality of this approximation, see section 2.3.2.

The individual vital rates are imported from the ForClim model and have the form ([25], Section 3.3)

$$G_{ind}(S, h) = w_H(h) \cdot w_L(S) \quad (1.16)$$

$$M_{ind}(S, h) = m_1 + m_2 \cdot \chi_{\{S \geq S_{mort}\}} \quad (1.17)$$

$$\Phi_{ind}(S) = \Phi_0 \cdot \chi_{\{S \leq S_{birth}\}} \quad (1.18)$$

with $m_i, \Phi_0, S_{mort}, S_{birth} > 0$; $w_H(h) \geq 0, w_L(S) \geq 0$ are continuous, and

$$\chi_{[a,b]}(x) := \begin{cases} 1 & : x \in [a, b], \\ 0 & : x \notin [a, b]. \end{cases}$$

$w_H(h)$ is the ‘‘optimal growth rate’’ of a tree, computed from a balance equation for biomass production vs. maintenance needs ([26]).

Moreover

$$w_H(h) > 0 \quad \forall h \in [h_0, H_{max}[, \quad (1.19)$$

$$w_H(h) = 0 \quad \forall h \geq H_{max}, \quad (1.20)$$

$$\partial_h w_H(H_{max}) = 0 \quad (1.21)$$

and there exists such a $H_{peak} \in]h_0, H_{max}[$ so that w_H is strictly increasing on $]h_0, H_{peak}[$ and strictly decreasing on $]H_{peak}, H_{max}[$.

w_L represents stress through overshadowing and is therefore strictly decreasing for $0 < S < S_{crit}$, zero for $S \geq S_{crit}$. This model’s dynamics have a

remarkable degree of similarity to those of ForClim. For a detailed discussion of the model by the authors, see [3]. Here I only note that also this model seems to have a single globally attractive steady-state independent of the initial conditions.

In their simulations, [3] have also tried to approximate $G(h, t)$, $M(h, t)$ and $\Phi(t)$ by $G_{ind}(E[S](h, t), h)$, $M_{ind}(E[S](h, t), h)$ and $\Phi_{ind}(E[S](h_0, t))$, respectively. This approximation proved too crude and provided a significantly poorer approximation of the patch dynamics, leading to extermination of some species present in the equilibrium state of both the patch model and the normal DisCForM model as described above.

This corresponds to the findings of [24] discussed in the previous section.

1.5 Motivation of the equations used in the present model

For the sake of simplicity I concentrate on one-species models. However, at least the existence and uniqueness results proved in this thesis should generalize directly to many-species systems.

All coefficient functions (including the expected number of newborns Φ per patch) are assumed to be a priori limited, which is biologically plausible. Also a certain least mortality rate is assumed:

$$\exists M_{min} > 0 : \quad M(h, t) \geq M_{min} \quad \forall (h, t) \quad (1.22)$$

I consider the second-order equation

$$\partial_t y = \partial_h^2 (D \cdot y) - \partial_h (G \cdot y) - M \cdot y \quad (1.23)$$

where G is the expected growth rate, and $2 \cdot D := M_2^G$ is the variance of the growth rate.

The total number of trees $n(t) := \int_{h_0}^{H_{max}} y(h, t) dh$ then satisfies

$$\partial_t n(t) = (\partial_h (D \cdot y) - G \cdot y) \Big|_{h_0}^{H_{max}} - \int_{h_0}^{H_{max}} y(h, t) M(h, t) dh \quad (1.24)$$

The change in the number of trees is thus due to mortality (the integral term) and flux through the boundaries. It is natural to interpret the flux through

the left boundary as sapling recruitment, thus arriving at the following left boundary condition :

$$G \cdot y - \partial_h(D \cdot y) = \Phi \quad (1.25)$$

The right boundary condition as well as the exact form of the coefficient functions will be discussed later in this section.

The reasons for considering a second-order equation are twofold:

Firstly, it should provide a better approximation of (1.1) than a first order equation, since it considers the variability of the growth rate at any given height, something a first-order model cannot do; therefore comparing the behaviour of our model with that of the much more extensively studied first-order models will provide insights on the importance of such variability;

Secondly, as there is always *some* natural variation in the growth rates (e.g. genetically predetermined), one can safely assume

$$\exists \delta > 0 : \quad D(h, t) \geq \delta \quad \forall (h, t) \quad (1.26)$$

Under this assumption (1.23) is parabolic, and the rich array of techniques that have been developed for study of parabolic equations can be applied. For instance, that the solutions of (1.23) are now as regular as the coefficients and the initial/boundary conditions allow, so all the derivatives in all directions can be defined in the classical sense (unlike the first order case).

The birth function Φ is assumed to depend on the momentary height distribution $y(\cdot, t)$ only. This ignores the fact that a sapling needs time to grow from a seed to the height h_0 .

In this thesis I assume that D and G depend only on h and $Y(h, t)$, Φ depends only on $Y(h_0, t)$; $Y(h, t)$ being the expected cumulative leaf area above height h :

$$Y(h, t) := \int_{h_0}^{H_{max}} y(h', t) \alpha(h') dh', \quad (1.27)$$

where $\alpha(h)$ is the crown area of a tree of height h . This does *not* imply that , say, $G(h, t) = G_{ind}(Y(h, t), h)$. As discussed in Section 1.3.2, the latter is a very poor approximation.

Instead, the ideal value of G (equal to which the expected value of $G_{ind}(S, h)$ with respect to a certain S -distribution) is approximated by some other function $\tilde{G}(Y(h, t), h)$. One way of constructing this \tilde{G} is discussed in Section

2.3.3. It should present no difficulty to generalize at least the existence and uniqueness results to the case where the coefficient functions are allowed to depend on several differently weighted integrals of $y(\cdot, t)$ instead of only Y , (say, the total number of trees higher than h : $n(h, t) := \int_h^{H_{max}} y(h', t) dh'$ or the variance of the cumulative basal area above h , which is under certain assumptions stated in Section 2.3.1 equal to $\int_h^{H_{max}} \alpha^2(h') y(h', t) dh'$), probably without changing any of the ideas involved in the proofs, provided the dependence would be smooth; but in this first work I restrict myself to a pure (Y, h) -dependency.

I assume that all coefficient functions are sufficiently *smooth* functions of their arguments Y and h (exact requirements are stated in Section 2.1). This is biologically justified, because even if some of the individual's vital rates are not smooth (compare [25]), computing their expected values with respect to the local light distribution will result in smooth coefficient functions.

The most natural right boundary condition seems to be

$$y|_{h=H_{max}} \equiv 0 \quad (1.28)$$

since H_{max} is defined as the maximum height attainable if one ignores the variance of the growth rate. However, our equation is parabolic, and that means that as long as there exist some trees with h near H_{max} , the variation of the growth rate will make some of them grow higher than H_{max} . In this case the above-mentioned boundary condition would correspond to a mystical force that instantly kills every tree whose height exceeds H_{max} .

A better condition would be

$$y|_{h=h_{max}} \equiv 0 \quad \text{for some } h_{max} \gg H_{max}, \quad (1.29)$$

since above H_{max} the combination of a steady least mortality rate with very low growth rates would result in a very fast exponential decay of the tree density y with increasing height, so the cutoff (1.29) at h_{max} would be the less noticeable, the bigger h_{max} is.

Mathematically, the above two possibilities are equivalent.

As h_{max} is now an arbitrary number which only has to be large enough, it is natural to consider the limit boundary condition

$$y(h, t) \xrightarrow{h \rightarrow \infty} 0 \quad \text{locally uniform in } t \quad (1.30)$$

Which of the two possibilities (1.29) and (1.30) one prefers depends largely on the context:

(1.30) is aesthetically more appealing, since no arbitrary cutoff needs to be made ; the obvious drawback is existence at any given moment of arbitrarily high trees; but because of the above-mentioned exponential decay the density of such trees is negligibly small; their influence can be further damped by letting $\alpha(h)$ (“the foliage area per tree of height h ”) go to zero for large h . if we do that, then (1.30) becomes only a matter of mathematical bookkeeping. One more reason for considering (1.30) is discussed in (3.2).

(1.29) is more appealing from a common sense point of view and more convenient to work with, especially if one is interested in numerical simulations. In the present thesis I will bear both options in mind.

In the Sections 2.1 and 2.2 I will for ease of notation set $h_0 = 0$. This has no biological relevance and is achieved through a variable change $h_{new} = h - h_0$.

Chapter 2

Results

2.1 Existence and uniqueness: The basis for further studies

2.1.1 Statement of results

Denote

$$\begin{aligned}\mathbb{R}_+ &:= [0, \infty[, \\ Q_T &:= \mathbb{R}_+ \times [0, T].\end{aligned}$$

For $l > 0$, $l \notin \mathbb{N}$ let

$$\begin{aligned}[l] &:= \sup\{x \in \mathbb{R} : x \leq l\} \\ |u|_{Q_T}^{(l)} &:= \langle u \rangle_{Q_T}^{(l)} + \sum_{j=0}^{[l]} \langle u \rangle_{Q_T}^{(j)} \\ \langle u \rangle_{Q_T}^{(j)} &:= \sum_{2r+s=j} |\partial_t^r \partial_h^s u|_{C^0(Q_T)} \\ \langle u \rangle_{Q_T}^{(l)} &:= \langle u \rangle_{h, Q_T}^{(l)} + \langle u \rangle_{t, Q_T}^{(l)} \\ \langle u \rangle_{h, Q_T}^{(l)} &:= \sum_{2r+s=[l]} \langle \partial_t^r \partial_h^s u \rangle_{h, Q_T}^{(l-[l])} \\ \langle u \rangle_{t, Q_T}^{(l)} &:= \sum_{0 \leq l-2r-s < 2} \langle \partial_t^r \partial_h^s u \rangle_{t, Q_T}^{\left(\frac{l-2r-s}{2}\right)}\end{aligned}$$

And, lastly, for $0 < \beta < 1$

$$\begin{aligned} \langle u \rangle_{h, Q_T}^{(\beta)} &:= \sup_{0 < |h_1 - h_2| \leq \rho} \frac{|u(h_1, t) - u(h_2, t)|}{|h_1 - h_2|^\beta} \\ \langle u \rangle_{t, Q_T}^{(\beta)} &:= \sup_{0 < |t_1 - t_2| \leq \rho} \frac{|u(h, t_1) - u(h, t_2)|}{|t_1 - t_2|^\beta} \end{aligned}$$

where ρ is some fixed positive constant.

Furthermore let

$$\begin{aligned} H^{l, \frac{l}{2}}(Q_T) &:= \{u : Q_T \rightarrow \mathbb{R} ; |u|^{(l)} < \infty\} \\ H_0^{l, \frac{l}{2}}(Q_T) &:= \{u \in H^{l, \frac{l}{2}}(Q_T) ; \partial_t^k u|_{t=0} \quad k = 0, \dots, \lfloor \frac{l}{2} \rfloor\} \end{aligned}$$

For these norms a product rule and a chain rule can be proved, having the form

$$\begin{aligned} |u \cdot v|^{(l)} &\leq |u|^{(l)} \cdot |v|^{(l)} \\ |F(u)|^{(l)} &\leq ([l] + 1)! |F|_{C^{[l]+1}} \cdot \left(|u|^{(l)}\right)^{[l]+1} \end{aligned}$$

for all $u, v \in H^{l, \frac{l}{2}}$, $F \in C^{[l]+1}$.

Moreover, for $f = f(g(h, t), h)$ we denote by $\partial_h f, \partial_t f$ the total derivatives of f with respect to h resp. t , and by f_g, f_h the partial derivatives, so that, for instance,

$$\partial_h f = f_h + f_g \partial_h g$$

We consider the existence and uniqueness of solutions

$$y = y(h, t) \in H^{l+2, \frac{l+2}{2}}(\mathbb{R}_+^2)$$

to the problem

$$\partial_t y = \mathcal{A}(y)y = A(Y, h)y \tag{2.1}$$

with boundary and initial conditions

$$\mathcal{B}(y)y|_{(0,t)} = \Phi(Y(0, t)) \tag{2.2}$$

and

$$y|_{t=0} = y_0 \quad (2.3)$$

Here, for any $f(h)$ we let $F = F(h)$ be defined by

$$F(h) := \int_h^\infty \alpha(h') f(h') dh', \quad (2.4)$$

where $\alpha = \alpha(h) \in C^2 \cap W^{2,1}(\mathbb{R}_+)$ is a fixed nonnegative weight function such that

$$h \cdot \alpha^{(j)}(h) \in L^1(\mathbb{R}_+) \quad \text{for } j = 0, 1, 2. \quad (2.5)$$

Moreover, we assume $0 \leq y_0 \in H^{l+2}(\mathbb{R}_+)$ for some $0 < l < 1$.

The operators \mathcal{A} and \mathcal{B} are given by

$$\mathcal{A}(f)g := \partial_h^2(D(F, h)g) - \partial_h(w(F, h)g) - m(F, h)g \quad (2.6)$$

$$\mathcal{B}(f)g := w(F, h)g - \partial_h(D(F, h)g) \quad (2.7)$$

Here $D, w, m : \mathbb{R}^2 \rightarrow \mathbb{R}_+$, $\Phi : \mathbb{R} \rightarrow \mathbb{R}_+$ are fixed functions satisfying

$$|D|_{C^4(\mathbb{R}^2)} + |w|_{C^3(\mathbb{R}^2)} + |m|_{C^2(\mathbb{R}^2)} + |\Phi|_{C^3(\mathbb{R})} < \infty \quad (2.8)$$

Furthermore, we assume

$$\exists \delta_1, \delta_2 \in \mathbb{R}_+ : \forall Y, h \quad 0 < \delta_1 \leq D(Y, h) \leq \delta_2, \quad (2.9)$$

$$\exists m_{\min} \in \mathbb{R}_+ : \forall Y, h \quad 0 < m_{\min} \leq m(Y, h). \quad (2.10)$$

All of the constants in the estimates below are allowed to depend tacitly on $\delta_1, \delta_2, |D|_{C^4(\mathbb{R}^2)}, |w|_{C^3(\mathbb{R}^2)}, |m|_{C^2(\mathbb{R}^2)}, |\Phi|_{C^3(\mathbb{R})}, |\alpha|_{C^2 \cap W^{2,1}(\mathbb{R}_+^2)}$ and $|h \cdot \partial^j \alpha(h)|_{L^1(\mathbb{R}_+)}$. We further assume the following compatibility condition:

$$\mathcal{B}(y_0)y_0|_{h=0} = \Phi(Y_0(0, 0)), \quad (2.11)$$

The main result of this section is

Theorem 1

Let $l > \frac{1}{2}$ and assume there exists a function $\mathcal{C} : H^{l+2}(\mathbb{R}_+) \rightarrow \mathbb{R}_+$ such that for every $T > 0$, for every solution $y \in H^{l+2, \frac{l+2}{2}}(Q_T)$ of (2.1)–(2.3) there holds

$$|y|_{C^0(Q_T)} \leq \mathcal{C}(y_0). \quad (2.12)$$

Also suppose

$$y_0(h)e^{\kappa h} \in H^{l+2}(\mathbb{R}_+) \quad \text{for some } \kappa > 0. \quad (2.13)$$

Then (2.1)–(2.3) has a unique solution $y \in H^{l+2, \frac{l+2}{2}}(\mathbb{R}_+^2)$.

That is, for exponentially decaying data problem (2.1)–(2.3) admits a global unique solution y , provided we can establish the a priori bound (2.12).

Note: (2.13) is satisfied, for instance, if y_0 has compact support.

For the proof of Theorem 1 we will need following lemmata:

Lemma 1 (Local existence)

There exists a positive constant τ^* depending on $|y_0|^{(l+2)}$ such that problem (2.1)–(2.3) has a unique solution $y \in H^{l+2, \frac{l+2}{2}}(Q_{\tau^*})$

Lemma 2 (Some a priori estimates)

Suppose $y \in H^{l+2, \frac{l}{2}+1}(Q_T)$ is a solution of (2.1)–(2.3) and suppose (2.13) is satisfied.

Then

- a) $y(h, t) \leq C(T)e^{-\kappa h} \quad \forall 0 \leq t \leq T$
- b) $y(h, t) > 0 \quad \forall 0 < t \leq T$
- c) $\exists \mathcal{Y}_{max} \in \mathbb{R}$ depending on y_0 but **not** on t or T , such that

$$|Y(\cdot, t)|_{C^0} \leq \mathcal{Y}_{max} \quad \forall t \leq T \quad (2.14)$$

For Lemma 3 we assume, as in Theorem 1, the existence of an a priori global C^0 -bound on y :

Lemma 3 Suppose that there exists a $\mathcal{C} : H^{l+2}(\mathbb{R}_+) \rightarrow \mathbb{R}_+$ such that

$$|y|_{C^0(Q_T)} \leq \mathcal{C} = \mathcal{C}(y_0). \quad (2.15)$$

for all $T > 0$, for all solutions $y \in H^{l+2, \frac{l+2}{2}}(Q_T)$ of the problem (2.1)–(2.3). Then there exist $K = K(\mathcal{C}) > 0$ and $\tilde{\tau} = \tilde{\tau}(\mathcal{C}) > 0$ such that if $\tau \leq \tilde{\tau}$,

$$|y|_{\mathbb{R}_+ \times [T, T+\tau]}^{(l+2)} \leq K \left(1 + |y(\cdot, T)|_{\mathbb{R}_+}^{(l+2)} \right) \quad (2.16)$$

for any solution $y \in H^{l+2, \frac{l+2}{2}}(Q_{T+\tau})$ of (2.1)–(2.3).

In the proofs of the lemmata we will make use of the following result from [27]:

Let Ω be a domain in \mathbb{R}^n , bounded or unbounded, let $Q_T := \Omega \times [0, T]$.

Consider the problem

$$\partial_t u - a_{11}(h, t) \partial_h^2 u + a_1(h, t) \partial_h u + a(h, t) u = f(h, t), \quad (2.17)$$

$$(b_1(h, t) \partial_h u + b(h, t) u)|_{\partial\Omega \times [0, T]} = \Phi(h, t), \quad (2.18)$$

$$u|_{t=0} = \phi(h). \quad (2.19)$$

Suppose a_{11}, b_1 satisfy

$$0 < \mu \leq a_{11}(h, t) \leq \nu \quad (2.20)$$

$$0 < \delta \leq b_1(h, t) \quad (2.21)$$

for some $\mu, \nu, \delta \in \mathbb{R}$, for all h, t .

Then the following theorem holds:

Theorem IV.5.3

Suppose $l > 0$ is a non-integral number, $\partial\Omega \in H^{l+2}$; a_{11}, a_1, a belong to the class $H^{l, \frac{l}{2}}(\bar{Q}_T)$, b_1, b belong to the class $H^{l+1, \frac{l+1}{2}}(\partial\Omega \times [0, T])$. Then for any $f \in H^{l, \frac{l}{2}}(\bar{Q}_T)$, $\phi \in H^{l+2}(\bar{\Omega})$, $\Phi \in H^{l+1, \frac{l+1}{2}}(\partial\Omega \times [0, T])$ satisfying the compatibility conditions of order $\lfloor \frac{l+1}{2} \rfloor$, problem (2.17)–(2.19) has a unique solution from the class $H^{l+2, \frac{l+2}{2}}(\bar{Q}_T)$ with

$$|u|_{Q_T}^{(l+2)} \leq C \left(|f|_{Q_T}^{(l)} + |\phi|_{\Omega}^{(l+2)} + |\Phi|_{\partial\Omega \times [0, T]}^{(l+1)} \right). \quad (2.22)$$

Since throughout this thesis we only consider $0 < l < 1$, we only define the compatibility condition of order 0. In our case $\Omega = [0, \infty[$, $\partial\Omega = 0$. The compatibility condition of order 0 can then be written as

$$(b_1 \partial_h u + b u)|_{(0,0)} = \Phi(0, 0). \quad (2.23)$$

In the problem (2.1)–(2.3),

$$a_{11}(h, t) = D(Y(h, t), h), \quad (2.24)$$

$$b_1(t) = D(Y(0, t), 0). \quad (2.25)$$

Therefore (2.20)–(2.21) follows from (2.9). The compatibility condition of order 0 is for (2.1)–(2.3) equivalent to (2.11).

Note that the theorem is applicable for both finite and infinite domains. For infinite domains the asymptotic bounds necessary for uniqueness are implied by $|y|^{(l+2)} < \infty$.

Proof of Theorem 1:

Global uniqueness: Consider two solutions y_1, y_2 of (2.1)–(2.3) both defined on an interval $[0, T[$. Consider the set

$$S := \{\bar{t} \in [0, T[: \forall 0 \leq t \leq \bar{t}, \forall h \quad y_1(h, t) = y_2(h, t)\} \quad (2.26)$$

S is open in $[0, T[$ because of Lemma 1, closed in $[0, T[$ because of continuity and $0 \in S$. We conclude $S = [0, T[$.

Global existence: Because of Lemma 1 a solution $y \in H^{l+2, \frac{l+2}{2}}$ of (2.1)–(2.3) exists on a maximal open time interval $[0, T[$, $T > 0$. If $T < \infty$, let $T_- := T - \tau$, where $\tau = \tilde{\tau}(\mathcal{C}(y_0))$ as in Lemma 3.

Then Lemma 3 yields

$$|y|_{Q_t}^{(l+2)} \leq K(1 + |y|_{Q_{T_-}}^{(l+2)}) \quad (2.27)$$

uniformly in $t < T$. By applying Lemma 1 on the problem (2.1)–(2.2) with the initial condition $y(\cdot, T - \frac{1}{2}\tau^*(K(1 + |y|_{Q_{T_-}}^{(l+2)})))$ we can extend y beyond T . ■

Similarly, we can prove global existence and uniqueness of solutions $y = y(h, t) \in H^{l+2, \frac{l+2}{2}}([0, h_{max}] \times \mathbb{R}_+)$ to (2.1)–(2.3) on a bounded height interval $[0, h_{max}]$ with boundary condition

$$y(h_{max}, t) = 0. \quad (2.28)$$

Here $h_{max} > 0$ is some positive constant, $Q_T := [0, h_{max}] \times [0, T]$ and the rest of the notations is identical to those of the previous section.

Theorem 2

Let $l > \frac{1}{2}$ and assume there exists a function $\mathcal{C} : H^{l+2}([0, h_{max}]) \rightarrow \mathbb{R}_+$ such that for every $T > 0$, for every solution $y \in H^{l+2, \frac{l+2}{2}}(Q_T)$ of (2.1)–(2.3), (2.28)

$$|y|_{C^0(Q_T)} \leq \mathcal{C}(y_0). \quad (2.29)$$

Then the problem (2.1)–(2.3), (2.28) has a unique solution y from the space $H^{l+2, \frac{l+2}{2}}([0, h_{max}] \times \mathbb{R}_+)$.

2.1.2 Proofs

Proof of Lemma 1:

For every $M > 0$ let

$$V_M(Q_\tau) := \left\{ y \in H^{l+2, \frac{l+2}{2}}(Q_\tau) \mid y|_{t=0} = y_0; \partial_t y|_{t=0} = \mathcal{A}(y_0)y_0; |y|_{Q_\tau}^{(l+2)} \leq M \right\} \quad (2.30)$$

Consider for $\tilde{y} \in V_M$ the problem

$$\frac{d}{dt}y = \mathcal{A}(\tilde{y})y, \quad (2.31)$$

$$y|_{t=0} = y_0, \quad (2.32)$$

$$\mathcal{B}(\tilde{y})y|_{(0,t)} = \Phi(\tilde{Y}(0, t)) \quad (2.33)$$

We claim that there exists an $M_0 = M_0(|y_0|^{(l+2)})$ such that for every $M \geq M_0$ there exists $\tau^*(M) > 0$ with the following properties:

- a) For all $\tilde{y} \in V_M$ the problem (2.31)–(2.33) has a unique solution $y \in V_M(Q_{\tau^*})$
- b) The solution operator L , which maps \tilde{y} onto the corresponding y , is a contraction, more precisely, for $\tilde{y}_1, \tilde{y}_2 \in V_M$ there holds

$$|L(\tilde{y}_1) - L(\tilde{y}_2)|_{Q_{\tau^*(M)}}^{(l+2)} \leq \frac{1}{2} |\tilde{y}_1 - \tilde{y}_2|_{Q_{\tau^*(M)}}^{(l+2)}. \quad (2.34)$$

- c) Problem (2.1)–(2.3) has a unique solution $y \in H^{l+2, \frac{l+2}{2}}(Q_{\tau^*(M_0)})$

Proof of a):

As all the coefficients depend on $\tilde{y} \in V_M(Q_\tau)$, this is a linear equation.

The compatibility condition of order 0 is fulfilled because of (2.11).

The coefficients a_{11}, a_1, a and b_1, b from Theorem IV.5.3. of [27] are as follows:

$$a_{11}(h, t) = D, \quad (2.35)$$

$$a_1(h, t) = w - 2\partial_h D, \quad (2.36)$$

$$a(h, t) = -\partial_h^2 D + \partial_h w + m, \quad (2.37)$$

right sides evaluated at $(\tilde{Y}(h, t), h)$.

$$b_1(t) = -D, \quad (2.38)$$

$$b(t) = w - \partial_h D, \quad (2.39)$$

right sides evaluated at $(\tilde{Y}(0, t), 0)$.

We first estimate $\left| \partial_h^k \tilde{Y} \right|^{(l)}$, $k = 0, 1, 2$ as follows:

$$\begin{aligned}\tilde{Y}(h, t) &= \int_h^\infty \alpha(h') \tilde{y}(h', t) dh' \\ \partial_h \tilde{Y}(h, t) &= -\alpha(h) \tilde{y}(h, t) \\ \partial_h^2 \tilde{Y}(h, t) &= -\alpha'(h) \tilde{y}(h, t) - \alpha(h) \partial_h \tilde{y}(h, t)\end{aligned}$$

Using the product rule we conclude

$$\begin{aligned}\left| \tilde{Y}(h, t) \right|^{(l)} &\leq C |\tilde{y}|^{(l)} \leq C \cdot M \\ \left| \partial_h \tilde{Y}(h, t) \right|^{(l)} &\leq C |\tilde{y}|^{(l)} \leq C \cdot M \\ \left| \partial_h^2 \tilde{Y}(h, t) \right|^{(l)} &\leq C |\tilde{y}|^{(l+1)} \leq C \cdot M\end{aligned}$$

To estimate the $H^{l, \frac{l}{2}}$ -norms of a_{11}, a_1, a and b_1, b , we combine the above estimates with the chain rule and the product rule, obtaining the bound

$$|a_{11}|_{Q_\tau}^{(l)} + |a_1|_{Q_\tau}^{(l)} + |a|_{Q_\tau}^{(l)} \leq C(M). \quad (2.40)$$

Similarly,

$$|b_1|_{[0, \tau]}^{(l+1)} + |b|_{[0, \tau]}^{(l+1)} \leq C(M). \quad (2.41)$$

Therefore a solution $y \in H^{l+2, \frac{l+2}{2}}(Q_\tau)$ exists for all τ, M , fulfilling

$$|y|_{Q_\tau}^{(l+2)} \leq C \left(|y_0|_{[0, \infty] \times \{0\}}^{(l+2)} + \left| \Phi(\tilde{Y}) \right|_{\{0\} \times [0, \tau]}^{(l+1)} \right) \quad (2.42)$$

(Theorem IV.5.3 from [27]).

For which M and τ is $|y|_{Q_\tau}^{(l+2)} \leq M$?

We estimate $\left| \Phi(\tilde{Y}) \right|_{\{0\} \times [0, \tau]}^{(l+1)}$ as follows:

$$\begin{aligned}\left| \Phi(\tilde{Y}) \right|^{(l+1)} &= |\Phi|_{C^0} + \left\langle \Phi(\tilde{Y}) \right\rangle_t^{(\frac{l+1}{2})} \leq \\ &\leq C |\Phi|_{C^1} \left(1 + \left| \partial_t \tilde{Y} \right|_{C^0} \right) \leq \\ &\leq C |\Phi|_{C^1} |\alpha|_{L^1} (1 + |\partial_t \tilde{y}|_{C^0}) \leq \\ &\leq C |\Phi|_{C^1} |\alpha|_{L^1} (1 + |\mathcal{A}(y_0) y_0|_{C^0} + \tau^{l/2} M)\end{aligned} \quad (2.43)$$

Inserting that in (2.42) gives us

$$|y|_{Q_\tau}^{(l+2)} \leq \tilde{C}(|y_0|^{(l+2)}) + C(M)\tau^{l/2} \quad (2.44)$$

so we define $M_0 := 2\tilde{C}(|y_0|^{(l+2)})$ and conclude that for all $M \geq M_0$ there exists a $\tau_1(M)$ such that

$$\forall \tau \leq \tau_1 \quad L : V_M(Q_\tau) \rightarrow V_M(Q_\tau)$$

Proof of b):

Let $\tau \leq \tau_1(M)$, $\tilde{u}, \tilde{v} \in V_M(Q_\tau)$,

$$u := L\tilde{u},$$

$$v := L\tilde{v}.$$

$w := u - v$ satisfies

$$\partial_t w - \mathcal{A}(\tilde{v})w = f(h, t), \quad (2.45)$$

$$\mathcal{B}(\tilde{v})w = \phi(t), \quad (2.46)$$

$$w(\cdot, 0) = 0. \quad (2.47)$$

Here

$$f = (\mathcal{A}(\tilde{u}) - \mathcal{A}(\tilde{v}))u, \quad (2.48)$$

$$\phi = -\Phi(\tilde{V}(0, t)) + \mathcal{B}(\tilde{v})u. \quad (2.49)$$

The compatibility condition of order 0 is satisfied since $\phi(0) = 0 = \mathcal{B}(\tilde{v})0$.

The $H^{l, \frac{l}{2}}$ -norms of the coefficients have been estimated in a). We must therefore only estimate $|f|_{Q_\tau}^{(l)}$ and $|\phi|_{[0, \tau]}^{(l+1)}$.

$$\begin{aligned} f(h, t) &= \partial_h^2 \left((D(\tilde{U}, h) - D(\tilde{V}, h))u \right) - \\ &\quad - \partial_h \left((w(\tilde{U}, h) - w(\tilde{V}, h))u \right) - \\ &\quad - (m(\tilde{U}, h) - m(\tilde{V}, h))u, \end{aligned} \quad (2.50)$$

$$\begin{aligned} \phi(h, t) &= \Phi(\tilde{U}, h) - \Phi(\tilde{V}, h) - \\ &\quad - \partial_h \left((D(\tilde{U}, h) - D(\tilde{V}, h))u \right) \Big|_{h=0} + \\ &\quad + (w(\tilde{U}, h) - w(\tilde{V}, h))u, \end{aligned} \quad (2.51)$$

Now for any $G \in C^1(\mathbb{R}^2)$

$$G(\tilde{U}, h) - G(\tilde{V}, h) = (\tilde{U} - \tilde{V}) \left(\int_0^1 G_1(s\tilde{U} + (1-s)\tilde{V}) ds \right), \quad (2.52)$$

where G_1 denotes the derivative of G with respect to the first argument. Therefor, similarly to a) we can use the chain rule and the product rule to prove

$$|f|^{(l)} \leq C(M) \sum_{j=0}^2 \left| \partial_h^j (\tilde{U} - \tilde{V}) \right|^{(l)} \leq C(M) \cdot |\tilde{u} - \tilde{v}|^{(l+1)}. \quad (2.53)$$

Similarly,

$$|\phi|^{(l+1)} \leq C(M) \sum_{j=0}^1 \left| \partial_h^j (\tilde{U} - \tilde{V}) \right|^{(l+1)} \leq C(M) \cdot |\tilde{u} - \tilde{v}|^{(l+1)}. \quad (2.54)$$

Then Theorem IV.5.3 of [27] yields

$$|u - v|_{Q_\tau}^{(l+2)} \leq C(M) |\tilde{u} - \tilde{v}|_{Q_\tau}^{(l+1)} \leq C(M) |\tilde{u} - \tilde{v}|_{Q_\tau}^{(2)} \leq \tau^{l/2} C(M) |\tilde{u} - \tilde{v}|_{Q_\tau}^{(l+2)} \quad (2.55)$$

The existence of a $\tau^* = \tau^*(M) \leq \tau_1(M)$ satisfying (2.34) is now obvious. Therefore, by the contraction theorem, (2.1)–(2.3) has a unique solution in $V_M(Q_{\tau^*(M)})$, which proves the existence part of the lemma by setting $M = M_0(y_0)$

c) Let y be the unique solution of (2.1)–(2.3) in $V_{M_0(y_0)}(Q_{\tau^*})$, where $\tau^* = \tau^*(M_0(y_0))$.

Let $y_1 \in H^{l+2, \frac{l+2}{2}}(Q_\tau)$ be another solution of (2.1)–(2.3). Let $t_{max} := \min(\tau, \tau^*)$,

$$S := \{\bar{t} \in [0, t_{max}] : \forall 0 \leq t \leq \bar{t}, \forall h \quad y_1(h, t) = y(h, t)\} \quad (2.56)$$

S is open in $[0, t_{max}]$ because of b), closed in $[0, t_{max}]$ because of continuity, $0 \in S$. We conclude $S = [0, t_{max}]$, and $y = y_1$ as long as both are defined. ■

Proof of Lemma 2:

a) Consider for $\eta \in H^{l+2, \frac{l+2}{2}}$ the linear problem

$$\partial_t \eta = e^{\kappa h} (\mathcal{A}(y) (e^{-\kappa h} \eta)) \quad (2.57)$$

$$\mathcal{B}(y)(e^{-\kappa h} \eta)|_{h=0} = \Phi(Y(0, t)) \quad (2.58)$$

$$\eta|_{t=0} = e^{\kappa h} \cdot y_0 \quad (2.59)$$

Since for all differentiable f

$$e^{\kappa h} (\partial_h (e^{-\kappa h} f)) = \partial_h f - \kappa f, \quad (2.60)$$

the coefficients of $\partial_h^2 \eta$ in (2.58) and of $\partial_h \eta$ in (2.59) are equal to $D(Y, h)$ and $D(Y, 0)$, respectively. The coefficients of lower h -derivatives of η are linear combinations of terms $\kappa^n \partial_h^j f(Y(h, t), h)$, where $n = 0, 1, 2$; $j = 0, 1$ and f is one of the coefficient functions D, w, m .

Therefore, as $y \in H^{l+2, \frac{l+2}{2}}$, all the coefficients in these equations are in $H^{l, \frac{l}{2}}$. Considering (2.9), Theorem IV.5.3 from [27] is again applicable, and a solution $\eta \in H^{l+2, \frac{l+2}{2}}$ exists with $|\eta|_{Q_T}^{(l+2)} \leq C(T)$. Now consider

$$\tilde{y} := e^{-\kappa h} \eta; \quad (2.61)$$

$\tilde{y} \in H^{l+2, \frac{l+2}{2}}$ and solves the same parabolic equation as y with the same boundary conditions, so we conclude $\tilde{y} = y$, yielding $\forall h \geq 0 \quad \forall 0 \leq t \leq T$

$$|y(h, t)| = |e^{-\kappa h} \eta(h, t)| \leq C(T) e^{-\kappa h} \quad (2.62)$$

$$|\partial_h y(h, t)| \leq C(T) e^{-\kappa h} \quad (2.63)$$

$$|\partial_h^2 y(h, t)| \leq C(T) e^{-\kappa h} \quad (2.64)$$

b)

We rewrite (2.1) as

$$\partial_t y = D(Y, h) \partial_h^2 y + \tilde{w}(Y, h, y) \partial_h y + \tilde{m}(Y, h, y) \cdot y, \quad (2.65)$$

with the definitions

$$\tilde{w} = -w + 2D_h - 3\alpha \cdot D_Y \cdot y \quad (2.66)$$

$$\begin{aligned} \tilde{m} = & -w_h + \alpha \cdot y \cdot w_Y - m + D_{hh} - \\ & -2D_{hY} \alpha \cdot y + D_{YY} \cdot (\alpha \cdot y)^2 - D_Y \partial_h \alpha \cdot y \end{aligned} \quad (2.67)$$

From a) we know $|y|_{C^0(Q_T)} \leq C(T)$. Therefore also

$$|\tilde{m}|_{C^0(Q_T)} \leq C(T) \quad (2.68)$$

Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be some smooth function, $1 \leq f \leq 2$.
We define

$$z(h, t) := e^{-C_1 t} \frac{y(h, t)}{f(h)} \quad (2.69)$$

or equivalently, $y(h, t) =: e^{C_2 t} \cdot f(h) \cdot z(h, t)$. Upon inserting that into the boundary condition, we obtain for $h = 0$:

$$D \cdot \partial_h z = \underbrace{\left(w - D_h + \alpha y D_Y - \frac{f'}{f} D \right)}_{a(h, t)} \cdot z - \frac{e^{-C_1 t}}{f} \Phi(Y).$$

We fix an f such that

$$a(h, t) \geq 1 \quad \forall (h, t) \in Q_T \quad (2.70)$$

(say $f(h) = 1 + \frac{\epsilon}{\epsilon+h}$, ϵ small enough) . This is possible as $|a(h, t)| \leq C(T)$.
The equation (2.65) can now be rewritten as

$$\begin{aligned} \partial_t z &= D \partial_h^2 z + \left(\tilde{w} + 2 \frac{\partial_h f}{f} D \right) \partial_h z + \\ &\quad + \underbrace{\left(\tilde{m} + \tilde{w} \cdot \frac{\partial_h f}{f} + D \cdot \frac{\partial_h^2 f}{f} - C_1 \right)}_{m_1(h, t)} \cdot z \end{aligned} \quad (2.71)$$

Since $f \in C^2([0, \infty])$ is fixed,

$$\exists C_1 : \quad \forall (h, t) \in Q_T \quad m_1(h, t) \leq -1 \quad (2.72)$$

Now we are ready to begin the proof.

Suppose

$$\inf_{0 \leq t \leq T} z(h, t) < 0 \quad (2.73)$$

Since (using a)) $z(h, t) \xrightarrow{h \rightarrow \infty} 0$ uniformly for $0 \leq t \leq T$,

$$\exists (\bar{h}, \bar{t}) : z(\bar{h}, \bar{t}) = \inf_{0 \leq t \leq T} z(h, t) \quad (2.74)$$

Case 1: $\bar{h} > 0$

In this case

$$\partial_{\bar{h}}^2 z(\bar{h}, \bar{t}) \geq 0, \quad \partial_{\bar{h}} z(\bar{h}, \bar{t}) = 0, \quad \partial_{\bar{t}} z(\bar{h}, \bar{t}) \leq 0. \quad (2.75)$$

But equation (2.71) gives

$$\partial_{\bar{t}} z(\bar{h}, \bar{t}) \geq m_1 \cdot z(\bar{h}, \bar{t}) \geq -z(\bar{h}, \bar{t}) > 0, \quad (2.76)$$

which is a contradiction.

Case 2: $\bar{h} = 0$

In this case the boundary condition implies at once

$$D \cdot \partial_{\bar{h}} z(\bar{h}, \bar{t}) \leq z(\bar{h}, \bar{t}) < 0, \quad (2.77)$$

contradicting the definition of (\bar{h}, \bar{t}) .

Conclusion: $z \geq 0$.

If we now suppose $z(\bar{h}, \bar{t}) = 0$ for some $\bar{t} > 0$, we first conclude $\bar{h} > 0$, using (2.70). Then the strong maximum principle gives us

$$y \equiv 0 \quad \forall (h, t) \in]0, \infty[\times]0, \bar{t}].$$

This and the continuity of y contradict $y|_{t=0} \not\equiv 0$, concluding the proof.

c) Let $g(t)$ be defined by

$$g(t) := \int_0^\infty y(h, t) dh \quad (2.78)$$

Then by (2.1),(2.2),(2.62) and (2.63)

$$g'(t) = - \int_0^\infty m(Y, h) y(h, t) dh + \Phi(Y) \leq -m_{\min} g(t) + |\Phi|_{C^0} \quad (2.79)$$

Therefore

$$g(t) \leq \frac{|\Phi|_{C^0}}{m_{\min}} + e^{-m_{\min} t} \left(g(0) - \frac{|\Phi|_{C^0}}{m_{\min}} \right) \leq \max \left(g(0), \frac{|\Phi|_{C^0}}{m_{\min}} \right) \quad (2.80)$$

Then

$$\begin{aligned} |Y(h, t)| &= \int_h^\infty \alpha(h) y(h, t) dh \leq |\alpha|_{C^0} \int_0^\infty y(h, t) dh \leq \\ &\leq |\alpha|_{C^0} \max \left(g(0), \frac{|\Phi|_{C^0}}{m_{\min}} \right) =: \mathcal{Y}_{\max} \end{aligned} \quad (2.81)$$

In the proof of Lemma 3 we will use

Lemma 4 *Under the assumptions of Lemma 3 , there exist global positive constants $\beta_0 = \beta_0(\mathcal{C})$, $C_1 = C_1(\mathcal{C})$, $\tau_1 = \tau_1(\mathcal{C})$ (independent of T and τ) such that $\forall h \in \mathbb{R}_+$; $t_1, t_2 \in [0, T + \tau]$:*

$$|t_1 - t_2| \leq \tau_1 \Rightarrow |Y(h, t_1) - Y(h, t_2)| \leq C_1 \cdot |t_1 - t_2|^{\beta_0} \quad (2.82)$$

Proof of Lemma 3 We first observe an important formula for the estimation of $\partial_t Y$, namely

$$\begin{aligned} \partial_t Y(h, t) &= \int_h^\infty \alpha(h) \partial_t y(h, t) dh = \\ &= -\alpha(h) (\partial_h(Dy) - wy) + yD\alpha' + \\ &\quad + \int_h^\infty (D\alpha'' + w\alpha' - m\alpha) y dh \end{aligned} \quad (2.83)$$

From this we can follow

$$|\partial_t Y(h, t)| \leq C |\partial_h y(h, t)| + C(\mathcal{C}) \quad (2.84)$$

When $h = 0$, (2.83) simplifies to

$$\begin{aligned} \frac{d}{dt} Y(0, t) &= \alpha(0)\Phi(Y(0, t)) + (yD\alpha')|_{h=0} + \\ &\quad + \int_0^\infty (D\alpha'' + w\alpha' - m\alpha) y dh \end{aligned} \quad (2.85)$$

We rewrite (2.1)–(2.3) as

$$\begin{aligned} \partial_t y - D(Y(h, T), h) \partial_h^2 y &= \underbrace{(-w + 2\partial_h D) \partial_h y + (-\partial_h w + \partial_h^2 D - m)y}_{f_1} + \\ &\quad + \underbrace{\left(D(Y(h, t), h) - D(Y(h, T), h) \right) \partial_h^2 y}_{f_2} \end{aligned} \quad (2.86)$$

$$\begin{aligned} D(Y(h, T), h) \partial_h y|_{h=0} &= \underbrace{(-\Phi(Y) + (w - \partial_h D)y)|_{h=0}}_{\phi_1} + \\ &\quad + \underbrace{\left(\left(D(Y(0, T), 0) - D(Y(0, t), 0) \right) \partial_h y \right)|_{h=0}}_{\phi_2} \end{aligned} \quad (2.87)$$

$$y|_{t=0} = y_0$$

where $D = D(Y(h, t), h)$, $w = w(Y(h, t), h)$, $m = m(Y(h, t), h)$ if no arguments are specified.

Now we check that $D(Y(h, T), h) \in H^{l, \frac{l}{2}}$. Namely, $D(Y, h) \in C^0$ according to (2.8), and

$$\begin{aligned} \langle D(Y(h, T), h) \rangle_h^{(l)} &\leq C \cdot |\partial_h D(Y(h, T), h)|_{C^0} = & (2.88) \\ &= C \cdot |D_h - \alpha \cdot y \cdot D_Y|_{C^0} \leq C \cdot |D|_{C^1} (1 + |\alpha|_{C^0} \mathcal{C}) \leq \\ &\leq C(\mathcal{C}), \end{aligned}$$

$$\langle D(Y(h, T), h) \rangle_t^{(\frac{l}{2})} = 0 \quad (2.89)$$

Therefore $|D(Y(h, T), h)|^{(l)} \leq C(\mathcal{C})$, and we can apply Theorem IV.5.3 from [27] upon the problem (2.87)–(2.88). It gives us

$$|y|_{\mathbb{R}_+ \times [T, T+\tau]}^{(l+2)} \leq C \left(|y|_{t=T}^{(l+2)} + |\phi_1 + \phi_2|_{\{0\} \times [T, T+\tau]}^{(l+1)} + |f_1 + f_2|_{\mathbb{R}_+ \times [T, T+\tau]}^{(l)} \right) \quad (2.90)$$

Now

$$|\phi_1|_{\{0\} \times [0, \tau]}^{(l+1)} = |\phi_1|_{C^0} + \langle \phi_1 \rangle_t^{(\frac{l+1}{2})} \leq |\phi_1|_{C^0} + C |\partial_t \phi_1|_{C^0}, \quad (2.91)$$

since the $h = 0$ boundary has no extension in the space direction.

The a priori C^0 bound on y implies an a priori C^0 bound on Y and therefore also on $|\phi_1|_{C^0}$. Now

$$\begin{aligned} |\partial_t \phi_1|_{C^0} &= |\partial_t [-\Phi(Y) + (w(Y, h) - D_h(Y, h) + \alpha \cdot y \cdot D_Y(Y, h)) y]|_{C^0} \leq \\ &\leq |\partial_t Y|_{C^0} (|\Phi'|_{C^0} + |w_Y|_{C^0} \mathcal{C} + |D_{hY}|_{C^0} \cdot \mathcal{C}^2 \cdot |\alpha|_{C^0}) + \\ &\quad + |\partial_t y|_{C^0} (|w|_{C^0} + |D_h|_{C^0} + 2|\alpha|_{C^0} \cdot |D_Y|_{C^0} \cdot \mathcal{C}) \leq \\ &\leq C(\mathcal{C}) \cdot (|\partial_t Y|_{C^0} + |\partial_t y|_{C^0}) \end{aligned} \quad (2.92)$$

Using (2.85) we obtain for $h = 0$

$$\begin{aligned} |\partial_t Y|_{C^0} &\leq |\alpha|_{C^0} |\Phi|_{C^0} + |D|_{C^0} \mathcal{C} |\alpha'|_{C^0} + \mathcal{C} |\alpha'' \cdot D + \alpha' \cdot w - m \cdot \alpha|_{L^1} \leq \\ &\leq C(\mathcal{C}), \end{aligned} \quad (2.93)$$

Therefore

$$|\partial_t \phi_1|_{C^0} \leq C(\mathcal{C})(1 + |y|^{(2)}), \quad (2.94)$$

and

$$|\phi_1|^{(l+1)} \leq C(\mathcal{C})(1 + |y|^{(2)}). \quad (2.95)$$

Now for the estimation of $|f_1|^{(l)}$.

$$\begin{aligned}
f_1 = & -w(Y, h) \cdot \partial_h y - 2D_h(Y, h)\partial_h y + 2\alpha D_Y(Y, h) \cdot y \cdot \partial_h y + \\
& + \left(-w_h(Y, h) + \alpha \cdot y \cdot w_Y(Y, h) - m + D_{hh}(Y, h) - \right. \\
& - 2D_{hY}(Y, h)\alpha \cdot y + D_{YY}(Y, h) \cdot (\alpha \cdot y)^2 - \\
& \left. - D_Y \alpha' \cdot y - D_Y \cdot \alpha \cdot \partial_h y \right) y
\end{aligned} \tag{2.96}$$

Summing it up, we get 3 types of terms:

$$(a) \quad G(Y, h) \cdot \partial_h y \tag{2.97}$$

$$(b) \quad G(Y, h) \cdot \partial_h y \cdot y \tag{2.98}$$

$$(c) \quad G(Y, h) \cdot y^n, \quad n = 1, 2, 3, \tag{2.99}$$

G being in each case some function $\mathbb{R}^2 \rightarrow \mathbb{R}$, $|G|_{C^1} \leq C$. One sees at once

$$|f_1|_{C^0} \leq C(\mathcal{C}) \cdot (1 + |\partial_h y|_{C^0}); \tag{2.100}$$

Furthermore,

$$\langle G \rangle_h^{(l)} \leq C \left| \frac{d}{dh} G \right|_{C^0} \leq C \cdot |G|_{C^1} (1 + |\alpha|_{C^0} \cdot \mathcal{C}) \leq C \tag{2.101}$$

and

$$\langle G \rangle_t^{(\frac{l}{2})} \leq |G|_{C^1} \langle Y \rangle_t^{(\frac{l}{2})} \leq |G|_{C^1} |\alpha|_{L^1} \langle y \rangle_t^{(\frac{l}{2})} \tag{2.102}$$

Using the interpolation inequalities (A.1), (A.2) proven in the Appendix A and the above we can now conclude

$$\langle f_1 \rangle_h^{(l)} \leq \left| \frac{d}{dh} f_1 \right|_{C^0} \leq C(\mathcal{C}) (|\partial_h^2 y|_{C^0} + |\partial_h y|_{C^0} + |\partial_h y|_{C^0}^2) \stackrel{(A.2)}{\leq} C(\mathcal{C}) |y|^{(2)} \tag{2.103}$$

and

$$\langle f_1 \rangle_t^{(\frac{l}{2})} \leq C (|\partial_h y|_t^{(\frac{l}{2})} + |\partial_h y|_{C^0} \cdot \langle y \rangle_t^{(\frac{l}{2})} + \langle y \rangle_t^{(\frac{l}{2})}) \stackrel{(A.1), (A.2)}{\leq} C(\mathcal{C}) |y|^{(2)} \tag{2.104}$$

Altogether

$$|f_1|^{(l)} \leq C(\mathcal{C}) (1 + |y|^{(2)}) \tag{2.105}$$

For the estimation of ϕ_2 and f_2 we first consider $D(Y(h, T), h) - D(Y(h, t), h)$:

$$\begin{aligned} |D(Y(h, T), h) - D(Y(h, t), h)| &\leq |D_Y|_{C^0} \cdot |Y(h, T) - Y(h, t)| \leq \\ &\leq C \cdot C_1 \cdot |T - t|^{\beta_0} \leq C \cdot C_1 \cdot \tau^{\beta_0} \end{aligned} \quad (2.106)$$

for $0 < \tau < \tau_1(\mathcal{C})$, with $\beta_0 > 0$, thanks to Lemma 4.

Therefore there exists a $\tau_2(\mathcal{C})$ such that for all τ with $0 < \tau < \tau_2(\mathcal{C})$, for all $t \in [T, T + \tau]$, for all h

$$|D(Y(h, T), h) - D(Y(h, t), h)| \leq \frac{1}{2} \quad (2.107)$$

Now

$$\begin{aligned} \langle D(Y(h, T), h) - D(Y(h, t), h) \rangle_h^{(l)} &\leq C \cdot |D_h - \alpha \cdot y \cdot D_Y|_{C^0} \leq \\ &\leq C \cdot |D|_{C^1} (1 + |\alpha|_{C^0} \mathcal{C}) \leq C(\mathcal{C}) \end{aligned} \quad (2.108)$$

$$\begin{aligned} \langle D(Y(h, T), h) - D(Y(h, t), h) \rangle_t^{(\frac{l}{2})} &\leq C \langle D(Y(h, t), h) \rangle_t^{(\frac{l}{2})} \stackrel{(A.1)}{\leq} \\ &\stackrel{(A.1)}{\leq} C \sqrt{|D|_{C^0} |\partial_t D(Y(h, t), h)|_{C^0}} \leq \\ &\leq C |D|_{C^1} \sqrt{|\partial_t Y(h, t)|_{C^0}} \stackrel{(2.84)}{\leq} \\ &\stackrel{(2.84)}{\leq} C \sqrt{C(\mathcal{C}) + |\alpha|_{C^0} |D|_{C^0} |\partial_h y|_{C^0}} \stackrel{(A.1)}{\leq} \\ &\stackrel{(A.1)}{\leq} C(\mathcal{C}) \sqrt{1 + |y|^{(1)}} \stackrel{(A.2)}{\leq} \\ &\stackrel{(A.2)}{\leq} C(\mathcal{C}) \left(1 + \left(|y|^{(2)} \right)^{\frac{1}{4}} \right) \end{aligned} \quad (2.109)$$

Therefore

$$\begin{aligned} |f_2|^{(l)} &\leq C(\mathcal{C}) \left(\left(1 + \left(|y|^{(2)} \right)^{\frac{1}{4}} \right) |\partial_h^2 y|_{C^0} + |\partial_h^2 y|^{(l)} \cdot \tau^{\beta_0} \right) \stackrel{(A.3)}{\leq} \\ &\stackrel{(A.3)}{\leq} C(\mathcal{C}) \left(1 + |y|^{(\frac{5}{2})} + |y|^{(l+2)} \cdot \tau^{\beta_0} \right) \end{aligned} \quad (2.110)$$

Now

$$|\phi_2|_{\{0\} \times [0, \tau]}^{(l+1)} = |\phi_2|_{C^0} + \langle \phi_2 \rangle_t^{(\frac{l+1}{2})}; \quad (2.111)$$

$$|\phi_2|_{C^0} \leq \frac{1}{2} |\partial_h y|_{C^0}; \quad (2.112)$$

$$\begin{aligned} \langle \phi_2 \rangle_t^{(\frac{l+1}{2})} &\leq \langle \partial_h y \rangle_t^{(\frac{l+1}{2})} |D(Y(0, T), 0) - D(Y(0, t), 0)|_{C^0} + \\ &\quad + |\partial_h y|_{C^0} \langle D(Y(0, t), 0) \rangle_t^{(\frac{l+1}{2})} \leq \\ &\leq C(\mathcal{C}) \left(|y|^{(l+2)} \tau^{\beta_0} + |\partial_h y|_{C^0} |D|_{C^1} |\partial_t Y|_{C^0} \right) \stackrel{(2.84)}{\leq} \\ &\stackrel{(2.84)}{\leq} C(\mathcal{C}) \left(|y|^{(l+2)} \tau^{\beta_0} + |\partial_h y|_{C^0}^2 + 1 \right) \leq \\ &\leq C(\mathcal{C}) \left(|y|^{(l+2)} \tau^{\beta_0} + |y|^{(2)} + 1 \right) \end{aligned} \quad (2.113)$$

Putting it all together into (2.90), we achieve

$$\begin{aligned} |y|_{\mathbb{R}_+ \times [T, T+\tau]}^{(l+2)} &\leq C(\mathcal{C}) \left(|y|_{t=T}^{(l+2)}|_{\mathbb{R}_+} + 1 + |y|^{(2\frac{1}{2})} + |y|^{(l+2)} \cdot \tau^{\beta_0} \right) \leq \\ &\leq C(\mathcal{C}) \left(|y|_{t=T}^{(l+2)}|_{\mathbb{R}_+} + 1 + |y|^{(l+2)} \cdot \left(\tau^{\beta_0} + \tau^{l/2 - \frac{1}{4}} \right) \right) \leq \\ &\leq \frac{1}{2} K(\mathcal{C}) \left(|y|_{t=T}^{(l+2)}|_{\mathbb{R}_+} + 1 \right) + \frac{1}{2} |y|_{\mathbb{R}_+ \times [T, T+\tau]}^{(l+2)} \end{aligned} \quad (2.114)$$

for all τ such that $0 < \tau \leq \tau_3(\mathcal{C})$. Here we have for the first and only time used $l > 2.5$.

We define

$$\tilde{\tau}(\mathcal{C}) := \min(\tau_1(\mathcal{C}), \tau_2(\mathcal{C}), \tau_3(\mathcal{C})) \quad (2.115)$$

and conclude $\forall 0 < \tau \leq \tilde{\tau}(\mathcal{C})$

$$|y|_{\mathbb{R}_+ \times [T, T+\tau]}^{(l+2)} \leq K(\mathcal{C}) \left(|y|_{t=T}^{(l+2)}|_{\mathbb{R}_+} + 1 \right). \quad (2.116)$$

Proof of Lemma 4 We use Theorem 2.3 from [28] to prove local Hölder estimates for Y_h and then integrate these to obtain global Hölder estimates for Y .

From the equation (2.83) we find

$$\begin{aligned} \frac{d}{dt}Y(h) &= D(Y, h) \cdot \partial_h^2 Y - \\ &\quad \underbrace{-\alpha(h) (\partial_h D - w) y + y D\alpha' + \int_h^\infty (D\alpha'' + w\alpha' - m\alpha) y dh}_{\tilde{a}(h,t)} \end{aligned} \quad (2.117)$$

Let

$$g(h) := |\alpha|_{L^1([h, \infty])} + |\alpha'|_{L^1([h, \infty])} + |\alpha''|_{L^1([h, \infty])} \quad (2.118)$$

Then

$$|\tilde{a}(h, t)| \leq C_2 \cdot g(h) \quad (2.119)$$

with $C_2 = C_2(\mathcal{C})$ independent of h .

(using $|\alpha^{(j)}(h)| \leq \int_h^\infty |\alpha^{(j+1)}(h')| dh' = |\alpha^{(j+1)}|_{L^1([h, \infty])}$)

Let

$$\rho_0(h) := \min(|h|, \rho_{max}) \quad (2.120)$$

$$R_{h,t}(\rho) :=]h - \rho, h + \rho[\times]t - \bar{\tau}\rho^2, t[\quad (2.121)$$

where $\rho_{max} > 0$, $\bar{\tau}$ are fixed constants.

Now fix a $\bar{h} \geq 0$ and consider

$$u(h, t) := \frac{1}{g(\bar{h} - \rho_0(\bar{h}))} Y(h, t) \quad (2.122)$$

which fulfills

$$\partial_t u = \underbrace{D(g(\bar{h} - \rho_0(\bar{h}))u, h)}_{a^{11}(u, h)} \partial_h^2 u + \frac{\tilde{a}(h, t)}{\underbrace{g(\bar{h} - \rho_0(\bar{h}))}_{a(h, t)}} \quad (2.123)$$

We want to apply Theorem 2.3 from [28] on $u|_{R_{(\bar{h}, \bar{\tau})}(\rho_0(\bar{h}))}$, for this we have to prove some estimates:

(i):

$$|u_h| = \frac{|\alpha(h)| |y|_{\mathcal{C}^0}}{g(\bar{h} - \rho_0(\bar{h}))} \leq \mathcal{C}$$

(ii):

$$1 \leq a^{11}(u, h) \leq \lambda$$

is fulfilled because of (2.9) (eventually after scaling h to achieve $\delta_1 = 1$)
(iii):

$$a_{u_x}^{11}(u, h) \equiv 0$$

(iv):

$$\begin{aligned} f(h, t) &:= \max \{ a_u^{11}(u, h), a_h^{11}(u, h), a(h, t) \} = \\ &= \max \left\{ D_h(h, t), g(\bar{h} - \rho_0(\bar{h})) D_Y(h, t), \frac{a(h, t)}{g(\bar{h} - \rho_0(\bar{h}))} \right\} \leq \\ &\leq |D_h|_{C^0} + g(0) |D_Y|_{C^0} + C_2 \leq \mu < \infty \end{aligned}$$

We note that all these estimates are (h, t) - independent.
Therefore Theorem 2.3 from [28] yields

$$\text{osc}_{R_{(\bar{h}, \bar{t})}(\rho_0(\bar{h}))} u_h \leq C \left(\frac{\rho}{\rho_0(\bar{h})} \right)^\delta \quad (2.124)$$

This is equivalent to

$$\text{osc}_{R_{(\bar{h}, \bar{t})}(\rho_0(\bar{h}))} Y_h \leq C \cdot g(\bar{h} - \rho_0(\bar{h})) \left(\frac{\rho}{\rho_0(\bar{h})} \right)^\delta \quad (2.125)$$

Here δ and C depend only on the right sides of the estimates (i)-(iv) and are therefore in our case global constants.

This implies for all $\tau > 0$, $\bar{t} \geq t_1 > 0$ such that $\tau \leq \rho_0(\bar{h})^2 \bar{\tau}$, $|t_1 - \bar{t}| \leq \tau$

$$|\partial_h Y(h, t_1) - \partial_h Y(h, \bar{t})| \leq C \cdot g(\bar{h} - \rho_0(\bar{h})) \left(\frac{\tau}{(\rho_0(\bar{h}))^2} \right)^{\delta/2} \quad (2.126)$$

Let τ_1 be defined by

$$\tau_0 = \bar{\tau} \cdot \rho_{max}, \quad (2.127)$$

Then for all t_1, t_2 such that $|t_1 - t_2| \leq \tau \leq \tau_0$

$$\begin{aligned}
|Y(h, t_1) - Y(h, t_2)| &\leq \int_0^\infty |\partial_h Y(h', t_1) - \partial_h Y(h', t_2)| dh' \leq \\
&\leq 2 \int_0^{\sqrt{\frac{\tau}{\bar{\tau}}}} |\alpha(h')y(h')| dh' + \\
&+ C \int_{\sqrt{\frac{\tau}{\bar{\tau}}}}^{\rho_{max}} g(\bar{h} - \rho_0(\bar{h})) \left(\frac{\tau}{(\rho_0(\bar{h}))^2} \right)^{\delta/2} dh' \\
&+ C \int_{\rho_{max}}^\infty g(\bar{h} - \rho_0(\bar{h})) \left(\frac{\tau}{(\rho_0(\bar{h}))^2} \right)^{\delta/2} dh' \\
&\leq (2\mathcal{C} |\alpha|_{C^0}) \sqrt{\frac{\tau}{\bar{\tau}}} + \\
&+ C \tau^{\delta/2} \int_0^{\rho_{max}} g(0) \left(\frac{1}{|h'|} \right)^\delta dh' + \\
&+ C \left(\frac{\tau}{\rho_{max}^2} \right)^{\delta/2} \int_0^\infty g(\bar{h} - \rho_{max}) dh' \leq \\
&\leq C \left(\tau^{\delta/2} + \sqrt{\frac{\tau}{\bar{\tau}}} \right) + \\
&+ C \left(\frac{\tau}{\rho_{max}^2} \right)^{\delta/2} \int_0^\infty (h' + \rho_{max}) (|\alpha(h')| + |\partial_h \alpha(h')| + |\partial_h^2 \alpha(h')|) dh' \leq \\
&\leq C \left(\tau^{\delta/2} + \sqrt{\frac{\tau}{\bar{\tau}}} \right) \leq C_1(\mathcal{C}) \tau^{\delta/2} \tag{2.128}
\end{aligned}$$

for $\tau \leq \tau_1$. Thus C_1 and τ_1 are defined; setting

$$\beta_0 := \delta/2 > 0 \tag{2.129}$$

concludes the proof. ■

Proof of Theorem 2

The proof is to a word identical to that of Theorem 1, using the analogues of Lemmata (1) –(4). These are again proved just as Lemmata (1) –(4):

In Lemma 1', one has to use a hybrid version of the Theorems 5.3 and 5.2

from Chapter IV of [27], leading to the estimate

$$|Q_\tau|^{(l+2)} \leq C \left(|y|_{t=0}^{(l+2)} + |\mathcal{B}y|_{h=0}^{(l+1)} + |y|_{h=h_{max}}^{(l+2)} \right) \quad (2.130)$$

But since $|0| = 0$ for *any* norm, the rest of the proof of Lemma 1' is identical to that of Lemma 1.

In Lemma 2', $y_0(h)e^{\kappa h} \in H^{l+2}$ is automatically true for all κ , and point a) makes no sense; but the proof of Point b) is again unchanged.

In point c), the additional term appears describing the flux through h_{max} , but since $y \geq 0$ and $y(h_{max}, t) \equiv 0$, this term only decreases the total number of trees and the estimate made there remains true.

Lemma 4 is basically a local result and therefore remains true.

The interpolation inequalities remain true for functions on finite intervals, only an additional term of the form $C(|f|_{C^0}, h_{max})$ appears on the right sides.

The proof of Lemma 3' proceeds exactly as that of Lemma 3 (taking into account the new form of the interpolation inequalities) and leads to the same estimate.

2.2 Analytical study of some model properties

2.2.1 Statement of results

Here we stop considering the most general case and turn to a special case which lends itself to a deeper treatment.

Namely, we restrict ourselves to the case

$$\Phi = \Phi(Y) > 0 \quad (2.131)$$

$$m = m(Y, h) > 0 \quad (2.132)$$

$$w = w_H(h) \cdot w_L(Y) \geq 0 \quad (2.133)$$

$$D = D(h) \quad (2.134)$$

such that for all Y, h

$$0 < m_{min} \leq m(Y, h) \leq m_{max} < \infty \quad (2.135)$$

$$\partial_Y m(Y, h) \geq 0 \quad (2.136)$$

$$w'_L(Y) \leq 0 \quad (2.137)$$

$$\Phi'(Y) \leq 0 \quad (2.138)$$

and

$$w_H(h) \geq \epsilon > 0 \quad (2.139)$$

for some $\epsilon \in \mathbb{R}$.

We write $D(h)$ as

$$D(h) = w_H(h) \cdot d(h). \quad (2.140)$$

and require in addition to (2.135)–(2.138)

$$w_H(h)d''(h) < \frac{1}{2}m_{min} \quad (2.141)$$

$$d'(0) < w_L(\mathcal{Y}_{max}) \quad (2.142)$$

where \mathcal{Y}_{max} is the a priori bound on Y from Lemma 2.

Both these conditions can be satisfied, for instance, by setting $D(h) = \epsilon D_0(h)$ with $\epsilon > 0$ small enough. Since the DisCForM model (that is discussed in Section 1.4) roughly corresponds to the limit $\epsilon = 0$, the case considered in this section is likely to produce behaviour most alike to that of DisCForM.

Now make the substitution

$$w_H(h) \cdot y(h, t) =: z(h, t). \quad (2.143)$$

and define Z by

$$Z(h, t) := \int_h^\infty \left(\frac{\alpha(h)}{w_H(h)} \right) z(h, t) dh = \int_h^\infty \alpha(h) y(h, t) dh = Y(h, t) \quad (2.144)$$

(This is strictly speaking inconsistent with (2.4), but it does simplify the notation.)

This transforms the equation into

$$\begin{aligned} \partial_t z &= w_H(h) \partial_h^2 (d(h) \cdot z) - w_H(h) \partial_h (w_L(Z) \cdot z) - m(Z, h) z \\ \text{resp.} \\ \partial_t z &= D(h) \partial_h^2 z - \left(w_H(h) \cdot w_L(Z) - 2w_H(h) d'(h) \right) \partial_h z - \\ &\quad - \left(m(Z, h) - w'_L(Z) \cdot \alpha(h) \cdot z - w_H(h) d''(h) \right) z \end{aligned} \quad (2.145)$$

and the left boundary condition into

$$w_L(Z) z - \partial_h (d(h) \cdot z) = \Phi(Z) \quad \text{for } h = 0 \quad (2.146)$$

resp.

$$\partial_h z = \frac{1}{d(0)} \left(\left(w_L(Z) - d'(0) \right) z - \Phi(Z) \right) \quad \text{for } h = 0 \quad (2.147)$$

Further, we consider the formulation of our problem with a Dirichlet right boundary condition on y for $h = h_{max} > 0$.

For this case one can prove

Theorem 3 (Asymptotical bounds) *Let $l > \frac{1}{2}$. Then*

a) *For every $z_0 \in H^{l+2, \frac{l+2}{2}}([0, h_{max}])$ there exists a solution $z(h, t) \in H^{l+2, \frac{l+2}{2}}([0, h_{max}] \times \mathbb{R}_+)$ of (2.145), (2.147) with $z(\cdot, t = 0) = z_0$.*

b) *There exist sequences $(z_+^i)_{i \in \mathbb{N}}$, $(z_-^i)_{i \in \mathbb{N}}$ such that for every $z(h, t) \in H^{l+2, \frac{l+2}{2}}([0, h_{max}] \times \mathbb{R}_+)$ solving (2.145), (2.147) there exist $(T^i)_{i \in \mathbb{N}}$ having the following property:*

$$z_-^i(h) \leq z(h, t) \leq z_+^i(h) \quad \forall t > T^i, 0 \leq h \leq h_{max}. \quad (2.148)$$

Moreover, $z_+^i, z_-^i \in H^{l+2}([0, h_{max}])$, $z_+^i(h_{max}) = 0$, $z_-^i(h_{max}) = 0$

$$z_+^i \leq z_+^j \quad \forall i > j \quad (2.149)$$

$$z_-^i \geq z_-^j \quad \forall i > j \quad (2.150)$$

$$z_-^i \leq z_+^j \quad \forall i, j. \quad (2.151)$$

All z_+^i, z_-^i are strictly positive for $h < h_{max}$, except $z_-^1 \equiv 0$.

The natural question now is whether these lower and upper bounding sequences converge to the same limit for $i \rightarrow \infty$, which would imply a globally attractive steady state independent of the initial conditions. However, that is by no means obvious.

Firstly, the lower and upper bounds constructed above certainly do *not* converge to the same limit, since the approximations of the elliptic operator \mathcal{A} used in the proof to construct the bounds are in general too coarse for that. One might, however, hope to achieve that convergence by using finer estimates; but already the simplistic version of the problem that is studied in the next section shows that additional restrictions on the coefficient functions will be needed to prove $|z_-^i - z_+^i| \rightarrow 0$ for $i \rightarrow \infty$ with the method used here.

The linear equation with nonlinear boundary condition

In this section we investigate the behaviour of $z_{+/-}^i$ as $i \rightarrow \infty$ in a simplistic model, where the only nonlinearity is that of the birth function.

Here we set

$$w_L(Y) \equiv 1 \quad (2.152)$$

$$m(Y, h) = m(h) \quad (2.153)$$

Then the equation (2.145) transforms into

$$\partial_t z = D(h) \partial_h^2 z - (w_H(h) - 2w_H(h)d'(h)) \partial_h z - (m(h) - w_H(h)d''(h)) z \quad (2.154)$$

and the boundary condition into

$$\underbrace{(1 - d'(0))}_{=:a} z - d(0) \partial_h z = \Phi(Z) \quad (2.155)$$

We assume, analogously to the previous section, that $a > 0$.

$\Phi(Z)$ is the same non-increasing positive function as in the previous section.

Let $z_*(h)$ be the time-independent solution of (2.154) with $z_*(0) = 1$, $z_*(h_{max}) = 0$. I define

$$-\partial_h z_*(0) =: \chi \quad (2.156)$$

$$Z_*(0) =: Z_* \quad (2.157)$$

(Lemma 5 proves that both these quantities are positive.)
Let furthermore $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be defined by

$$G(\gamma) := \frac{\Phi(\mathcal{Z}_*\gamma)}{a + \chi d(0)}. \quad (2.158)$$

Then $G(\gamma)$ is non-increasing (because Φ is) and there exists a unique solution γ_* of $G(\gamma) = \gamma$.

$\gamma_* z_*(h)$ is then a steady-state solution of (2.154),(2.155).

Theorem 4

$|z_-^i - z_+^i|_{C^0} \rightarrow 0$ for $i \rightarrow \infty$ if and only if the equation

$$\gamma = G(G(\gamma)) \quad (2.159)$$

has γ_* as its only solution . If this is the case , $\gamma_* z_*(h)$ is the globally attractive equilibrium solution of the problem (2.154),(2.155), independent of the initial conditions.

2.2.2 Proofs

Lemma 5 Let $W(h) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, $M(h) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}$. Then $\exists ! z : [0, h_{max}] \rightarrow \mathbb{R}_+$, $z \in C^2$, $z(h_{max}) = 0$, $z(0) = 1$ solving

$$0 = D(h)\partial_h^2 z - W(h)\partial_h z - M(h)z \quad (2.160)$$

Furthermore $\exists \kappa > 0 : \partial_h z \leq -\kappa < 0 \forall h \in [0, h_{max}]$

Proof of Lemma 5

Solve the ODE 2.160 starting from $\bar{z}(h_{max}) = 0$, $\partial_h \bar{z}(h_{max}) = -1$. Since this equation is linear, the solution \bar{z} exists for $h \in [0, h_{max}]$, $\bar{z}(0) =: \gamma$. We first prove that $\partial_h \bar{z} < 0 \forall h \in [0, h_{max}]$:

Suppose

$$\partial_h \bar{z}|_{h=\bar{h}} > 0 \quad (2.161)$$

for some $\bar{h} \in [0, h_{max}]$. Then for some $\tilde{h} \in]\bar{h}, h_{max}[$ $\partial_h \bar{z}$ changes sign from positive to negative, producing a local maximum which contradicts the weak maximum principle. Therefore we conclude

$$\partial_h \bar{z} \leq 0 \quad \forall h \in]0, h_{max}], \quad (2.162)$$

which implies $\bar{z} > 0 \quad \forall h < h_{max}$, in particular $\gamma > 0$.

Since \bar{z} is nonincreasing, the strong maximum principle gives us $\partial_h \bar{z} < 0 \quad \forall h \in [0, h_{max}]$, therefore, as $[0, h_{max}]$ is compact,

$$\exists \kappa > 0 : \quad \partial_h \bar{z} \leq -\kappa \gamma < 0 \quad \forall h \in [0, h_{max}] \quad (2.163)$$

Defining $z := \frac{1}{\gamma} \bar{z}$ completes the proof.

Proof of Theorem 3

We first assume the existence of

$$z(h, t) \in H^{l+2, \frac{l+2}{2}}([0, h_{max}] \times [0, T])$$

solving (2.145), (2.147) and seek to prove a priori C^0 -bounds for all its period of existence.

We choose a δ such that

$$\frac{1}{2} m_{min} > \delta > 0 \quad (2.164)$$

and define

$$\begin{aligned} \mathcal{A}_1(f)g &:= -D(h) \partial_h^2 g + \left(w_H(h) \cdot w_L(F) - 2w_H(h) d'(h) \right) \partial_h g + \\ &\quad + \left(m(F, h) - w_H(h) d''(h) \right) g \end{aligned} \quad (2.165)$$

$$\mathcal{A}_2(f)g := \left(-w_H(h) w_L'(F) \cdot \alpha(h) \cdot f \right) g \quad (2.166)$$

$$\mathcal{A}_{+, \delta}(f)g := -\delta g \quad (2.167)$$

$$\mathcal{A}_{-, \delta}(f)g := \left(w_H(h) |w_L'|_{C^0} \cdot \alpha(h) \cdot f + \delta \right) g \quad (2.168)$$

Then $z(h, t)$ fulfills

$$\partial_t z + \mathcal{A}_1(z)z + \mathcal{A}_2(z)z = 0 \quad (2.169)$$

We will construct the bounds iteratively.

Until otherwise stated we consider δ as fixed. Let $i \geq 1$

Suppose $\exists T_\delta^{i-1}, t_-^i, z_{-, \delta}^i(h) : \quad \forall t \geq T_\delta^{i-1} + t_-^i, \forall h$

$$z(h, t) \geq z_{-, \delta}^i(h) \quad (2.170)$$

(This is true for $i=1$ if we set $T_\delta^0 := 0, z_{-, \delta}^1(h) := 0, t_-^1 := 0$.) Let $\tilde{z}_{+, \delta}^i$ be the solution of

$$\mathcal{A}_1(z_{-, \delta}^i) \tilde{z}_{+, \delta}^i + \mathcal{A}_{+, \delta}(z_{-, \delta}^i) \tilde{z}_{+, \delta}^i = 0 \quad (2.171)$$

fulfilling $\tilde{z}_{+, \delta}^i(0) = 1, \tilde{z}_{+, \delta}^i(h_{max}) = 0$.

The existence of such a $\tilde{z}_{+, \delta}^i$ is proved in Lemma 5, which is applicable because of (2.141) and (2.164).

Let $\chi_{+, \delta}^i$ be defined by

$$\chi_{+, \delta}^i := -\partial_h \tilde{z}_{+, \delta}^i|_{h=0} \quad (2.172)$$

Lemma 5 proves $\chi_{+, \delta}^i > 0$. We take some $\gamma \in C^1([T_\delta^{i-1} + t_-, \infty[, \mathbb{R}_+)$ such that

$$0 \leq \gamma'(t) \leq -\delta\gamma(t) \quad (2.173)$$

and define

$$\mathcal{Z}(h, t) := \gamma(t) \tilde{z}_{+, \delta}^i(h) \quad (2.174)$$

By taking $\gamma(T_\delta^{i-1} + t_-)$ big enough we can achieve

$$\mathcal{Z}(h, T_\delta^{i-1} + t_-) = \gamma(T_\delta^{i-1} + t_-) \tilde{z}_{+, \delta}^i(h) \geq z(h, T_\delta^{i-1} + t_-) \quad \forall h. \quad (2.175)$$

Furthermore

$$\begin{aligned} \partial_t(z - \mathcal{Z}) + \mathcal{A}_1(z)(z - \mathcal{Z}) &\leq -\mathcal{A}_2(z)z + \delta\mathcal{Z} - \mathcal{A}_1(z)\mathcal{Z} = \\ &\leq w_H(h)w'_L(Z)\alpha(h)z^2 + (\mathcal{A}_1(z_{-, \delta}^i) - \mathcal{A}_1(z))\mathcal{Z} \stackrel{(2.137)}{\leq} \\ &\stackrel{(2.137)}{\leq} w_H(h)(w_L(Z_{-, \delta}^i) - w_L(Z))\partial_h\mathcal{Z} + \\ &\quad + (m(Z_{-, \delta}^i, h) - m(Z, h))\mathcal{Z} \leq 0, \end{aligned} \quad (2.176)$$

and

$$(z - \mathcal{Z})|_{h=h_{max}} \equiv 0 \quad (2.177)$$

Proposition 1 *If* $\exists T > 0 : \forall 0 \leq t \leq T$

$$(z - \mathcal{Z})(0, t) = 0 \quad \text{implies} \quad \partial_h(z - \mathcal{Z})|_{(0,t)} > 0 \quad (2.178)$$

then for all $0 \leq h \leq h_{max}, 0 \leq t \leq T$

$$(z - \mathcal{Z})(h, t) \leq 0 \quad (2.179)$$

Proof of Proposition 1.

Case 1: $(z - \mathcal{Z})(0, t) \leq 0 \quad \forall 0 \leq t \leq T$

In this case the weak maximum principle concludes the proof.

Case 2: $\exists 0 < \tilde{t} \leq T : (z - \mathcal{Z})(0, \tilde{t}) > 0$

$\Rightarrow \exists 0 < \bar{t} \leq T : (z - \mathcal{Z})(0, t) < 0 \quad \forall 0 \leq t < \bar{t}, \quad (z - \mathcal{Z})(0, \bar{t}) = 0.$

Then

$$\partial_t(z - \mathcal{Z})|_{(0,\bar{t})} \geq 0. \quad (2.180)$$

The maximum principle again implies

$$(z - \mathcal{Z})(h, t) \leq 0 \quad \forall (h, t) \in [0, h_{max}] \times [0, \bar{t}].$$

So $\partial_h(z - \mathcal{Z})|_{(0,\bar{t})} > 0$ leads at once to contradiction. \blacksquare

Proof of Theorem 3 continued.

To apply Proposition 1 for obtaining bounds on z we have to construct a $\gamma(t)$ such that the assumption (2.178) is fulfilled for all $t \geq T_\delta^{i-1} + t_-^i$.

Now if $z(0, t) = \mathcal{Z}(0, t) = \gamma(t)$, then

$$\begin{aligned} \partial_h(z - \mathcal{Z})|_{h=0} &\stackrel{(2.147)}{=} \frac{1}{d(0)} \left((w_L(Z) - d'(0))z - \Phi(Z) \right) - \partial_h \mathcal{Z}|_{h=0} \geq \\ &\geq \frac{1}{d(0)} \left((w_L(Z(0, t)) - d'(0))\gamma(t) - \Phi(Z_{-, \delta}^i(0)) \right) + \chi_{+, \delta}^i \gamma(t) \geq \\ &\geq \frac{1}{d(0)} \left((w_L(\min(\mathcal{Y}_{max}, \gamma(t)\tilde{Z}_{+, \delta}^i(0))) - d'(0) + d(0)\chi_{+, \delta}^i) \gamma(t) - \Phi(Z_{-, \delta}^i(0)) \right) \end{aligned}$$

Therefore a sufficient condition for (2.178) is

$$\left(w_L(\min(\mathcal{Y}_{max}, \gamma(t)\tilde{Z}_{+, \delta}^i(0))) - d'(0) + d(0)\chi_{+, \delta}^i \right) \gamma(t) > \Phi(Z_{-, \delta}^i(0)) \quad (2.181)$$

for all $t \geq T_\delta^{i-1} + t_-^i$.

Because of (2.142) , (2.181) is true if

$$\gamma(t) > \gamma_{+, \delta}^i \quad \forall t \geq T_\delta^{i-1} + t_-^i, \quad (2.182)$$

where $\gamma_{+, \delta}^i$ is the biggest solution of

$$\left(w_L(\min(\mathcal{Y}_{max}, \gamma \tilde{Z}_{+, \delta}^i(0))) - d'(0) + d(0)\chi_{+, \delta}^i \right) \gamma = \Phi(Z_{-, \delta}^i(0)). \quad (2.183)$$

Summing it all up, we have proved the following:

For every $\gamma(t)$ satisfying for all $t \geq T_\delta^{i-1} + t_-^i$ (2.173), (2.175) and (2.182) there holds

$$0 < z(h, t) \leq \gamma(t) \tilde{z}_{+, \delta}^i(h) \quad (2.184)$$

Proof of a): Let $i = 1$.

We choose a constant $\Gamma > \gamma_{+, \delta}^i$ such that (2.175) holds. Now for $\gamma(t) \equiv \Gamma$ the estimate (2.184) yields

$$|z(h, t)| \leq \Gamma \cdot \tilde{z}_{+, \delta}^i \stackrel{L.5}{\leq} \Gamma. \quad (2.185)$$

Theorem 1 then proves a).

Proof of b): Let

$$t_+^i := -\frac{1}{\delta} \ln \left(\frac{\gamma_{+, \delta}^i}{\Gamma} \right) + 1 \quad (2.186)$$

$$T_\delta^i := T_\delta^{i-1} + t_-^i + t_+^i \quad (2.187)$$

Then there exists a $\gamma \in C^1([T_\delta^{i-1} + t_-^i, \infty[, \mathbb{R}_+)$ fulfilling (2.173) such that

$$\gamma(T_\delta^{i-1} + t_-^i) = \Gamma \quad (2.188)$$

$$\gamma(t) \equiv \gamma_{+, \delta}^i \quad \forall t \geq T_\delta^i \quad (2.189)$$

Let $\epsilon > 0$,

$$\gamma_\epsilon(t) := \gamma(t) + \epsilon \quad (2.190)$$

$$\mathcal{Z}_\epsilon(h, t) := \gamma_\epsilon(t) \tilde{z}_{+, \delta}^i(h) \quad (2.191)$$

Then $\gamma_\epsilon(t)$ satisfies (2.173), $\gamma_\epsilon(t) > \gamma_{+, \delta}^i$ for all t , therefore (2.181) is satisfied for all t .

Proposition 1 then proves

$$\forall t \geq T_\delta^{i-1} + t_-^i, \forall h \quad z(h, t) \leq \mathcal{Z}_\epsilon(h, t). \quad (2.192)$$

As this is true for all $\epsilon > 0$, we conclude

$$\forall t \geq T_\delta^{i-1} + t_-^i, \forall h \quad z(h, t) \leq \mathcal{Z}(h, t). \quad (2.193)$$

Therefore

$$\forall t \geq T_\delta^i, \forall h \quad z(h, t) \leq \gamma_{+, \delta}^i \tilde{z}_{+, \delta}^i =: z_{+, \delta}^i \quad (2.194)$$

Along the same lines we define $\tilde{z}_{-, \delta}^{i+1}$ as the solution of

$$\mathcal{A}_1(z_{+, \delta}^i) \tilde{z}_{-, \delta}^{i+1} + \mathcal{A}_{-, \delta}(z_{+, \delta}^i) \tilde{z}_{-, \delta}^{i+1} = 0 \quad (2.195)$$

fulfilling $\tilde{z}_{-, \delta}^{i+1}(0) = 1$, $\tilde{z}_{-, \delta}^{i+1}(h_{max}) = 0$. We proceed to define

$$0 < \chi_{-, \delta}^{i+1} := -\partial_h \tilde{z}_{-, \delta}^{i+1}|_{h=0} \quad (2.196)$$

and $\gamma_{-, \delta}^{i+1}$ as the smallest solution of

$$\left(w_L(\gamma \tilde{Z}_{-, \delta}^{i+1}(0)) - d'(0) + d(0) \chi_{+, \delta}^i \right) \gamma = \Phi(Z_{+, \delta}^i(0)) \quad (2.197)$$

The we prove quite analogously to the above, that

$$\exists t_-^{i+1} : \forall t \geq T_\delta^i + t_-^{i+1} \quad z(h, t) \geq \gamma_{-, \delta}^{i+1} \tilde{z}_{-, \delta}^{i+1} =: z_{-, \delta}^{i+1} \quad (2.198)$$

Thus the induction step is complete, and for every δ such that $\frac{1}{2}m_{min} > \delta > 0$ we have constructed a sequence of asymptotic lower and upper bounds. We now take some sequence $\delta_n \xrightarrow{n \rightarrow \infty} 0$ satisfying $\frac{1}{2}m_{min} > \delta_n > 0$ for all n and define

$$z_+^i := z_{+, \delta_i}^i \quad (2.199)$$

$$z_-^i := z_{-, \delta_i}^i \quad (2.200)$$

$$T^i := T_{\delta_i}^i \quad (2.201)$$

To finish the proof of Theorem 3 we only need some monotonicity results, which we prove in Lemma 6. \blacksquare

In Lemma 6 certain monotonicity properties of $\tilde{z}_{+,\delta}^i$, $\tilde{z}_{-,\delta}^i$, $\gamma_{+,\delta}^i$, $\gamma_{-,\delta}^i$, $\chi_{+,\delta}^i$, $\chi_{-,\delta}^i$ are proved for $\frac{1}{2}m_{\min} > \delta \geq 0$. In the case $\delta = 0$ these sequences can be defined by iteration just as above. They don't yield any asymptotic bounds, but will be useful later as a limiting case.

Lemma 6 *The sequences $\tilde{z}_{+,\delta}^i$, $\tilde{z}_{-,\delta}^i$, $\gamma_{+,\delta}^i$, $\gamma_{-,\delta}^i$, $\chi_{+,\delta}^i$, $\chi_{-,\delta}^i$ depend on i and δ in a monotone manner. More precisely, for all $h \in [0, h_{\max}[$, for all δ_1, δ_2 satisfying $\frac{1}{2}m_{\min} > \delta_1 > \delta_2 \geq 0$, for all $i > 0$ there hold the following inequalities: Monotonicity in i :*

$$\tilde{z}_{-,\delta_2}^i \leq \tilde{z}_{-,\delta_2}^{i+1} \leq \tilde{z}_{+,\delta_2}^{i+1} \leq \tilde{z}_{+,\delta_2}^i \quad (2.202)$$

$$\gamma_{-,\delta_2}^i \leq \gamma_{-,\delta_2}^{i+1} \leq \gamma_{+,\delta_2}^{i+1} \leq \gamma_{+,\delta_2}^i \quad (2.203)$$

$$\chi_{-,\delta_2}^i \geq \chi_{-,\delta_2}^{i+1} \geq \chi_{+,\delta_2}^{i+1} \geq \chi_{+,\delta_2}^i \quad (2.204)$$

Monotonicity in δ :

$$\tilde{z}_{-,\delta_1}^i < \tilde{z}_{-,\delta_2}^i < \tilde{z}_{+,\delta_2}^i < \tilde{z}_{+,\delta_1}^i \quad (2.205)$$

$$\gamma_{-,\delta_1}^i < \gamma_{-,\delta_2}^i < \gamma_{+,\delta_2}^i < \gamma_{+,\delta_1}^i \quad (2.206)$$

$$\chi_{-,\delta_1}^i > \chi_{-,\delta_2}^i > \chi_{+,\delta_2}^i > \chi_{+,\delta_1}^i \quad (2.207)$$

Proof: The proofs are all entirely analogous; We will therefore prove only the right and left inequalities of (2.202)–(2.204). The index δ_1 is omitted everywhere for ease of notation.

The proof proceeds by induction in i :

We know $0 \equiv z_-^1 \leq z_-^2$. Suppose $z_-^{i+1} \leq z_-^i \quad \forall h \in [0, h_{\max}[$. Then

$$\begin{aligned} & (\mathcal{A}_1(z_-^i) + \mathcal{A}_+(z_-^i)) (\tilde{z}_+^{i+1} - \tilde{z}_+^i) = \\ & = (\mathcal{A}_1(z_-^i) + \mathcal{A}_+(z_-^i) - \mathcal{A}_1(z_-^{i+1}) + \mathcal{A}_+(z_-^{i+1})) \tilde{z}_+^{i+1} = \\ & = w_H(h) (w_L(Z_-^i) - w_L(Z_-^{i+1})) \partial_h \tilde{z}_+^{i+1} + \\ & \quad + (m(Z_-^i, h) - m(Z_-^{i+1}, h)) \tilde{z}_+^{i+1} \leq 0 \end{aligned}$$

Therefore

$$\tilde{z}_+^{i+1}(h) \leq \tilde{z}_+^i(h) \quad (2.208)$$

$$\chi_+^{i+1} \geq \chi_+^i \quad (2.209)$$

$$\tilde{Z}_+^{i+1}(0) \leq \tilde{Z}_+^i(0) \quad (2.210)$$

and so

$$\begin{aligned}
& \left(w_L(\min(\mathcal{Y}_{max}, \gamma_+^i \tilde{Z}_+^{i+1}(0))) - d'(0) + d(0)\chi_+^{i+1} \right) \gamma_+^i \geq & (2.211) \\
& \geq \left(w_L(\min(\mathcal{Y}_{max}, \gamma_+^i \tilde{Z}_+^i(0))) - d'(0) + d(0)\chi_+^i \right) \gamma_+^i = \\
& = \Phi(Z_-^i(0)) \\
& \geq \Phi(Z_-^{i+1}(0))
\end{aligned}$$

We conclude

$$\gamma_+^{i+1} \leq \gamma_+^i \quad (2.212)$$

The other half of the induction step, namely

$$\tilde{z}_-^{i+1}(h) \leq \tilde{z}_-^{i+2}(h) \quad (2.213)$$

$$\chi_-^{i+1} \geq \chi_-^{i+2} \quad (2.214)$$

$$\gamma_-^{i+1} \leq \gamma_-^{i+2} \quad (2.215)$$

is proven exactly in the same manner.

Note: if at least one of m, w_L, Φ is strictly monotone with respect to Z , the inequalities of this lemma all become strict. Proof by induction, propagating the strict inequalities $z_-^2, z_+^1 > z_-^1 \equiv 0$ ■

Proof of Theorem 4 The sequences z_+^i, z_-^i have been defined in the proof of Theorem 3.

However, since the equation is now linear,

$$z_{+, \delta}^i = \gamma_{+, \delta}^i \tilde{z}_{+, \delta}, \quad (2.216)$$

$$z_{-, \delta}^i = \gamma_{-, \delta}^i \tilde{z}_{+, \delta} \quad (2.217)$$

with $\tilde{z}_{+, \delta}, \tilde{z}_{-, \delta}$ independent of i . Correspondingly

$$\partial_h \tilde{z}_{+, \delta}|_{h=0} = \chi_{+, \delta} > 0 \quad (2.218)$$

$$\partial_h \tilde{z}_{-, \delta}|_{h=0} = \chi_{-, \delta} > 0 \quad (2.219)$$

are independent of i . Furthermore

$$|\chi_{+, \delta} - \chi_{-, \delta}| \xrightarrow{\delta \rightarrow 0} 0, \quad (2.220)$$

$$|\tilde{z}_{+, \delta} - \tilde{z}_{-, \delta}|_{C^0} \xrightarrow{\delta \rightarrow 0} 0. \quad (2.221)$$

because $\tilde{z}_{+,\delta}, \tilde{z}_{-,\delta}$ are solutions of a linear ODE with coefficients smoothly dependent on δ .

The proof proceeds in three steps:

Step 1

$$|z_+^i - z_-^i|_{C^0} \xrightarrow{i \rightarrow \infty} 0 \quad \Leftrightarrow \quad \inf_{\delta > 0, i \in \mathbb{N}} \gamma_{+,\delta}^i = \sup_{\delta > 0, i \in \mathbb{N}} \gamma_{-,\delta}^i \quad (2.222)$$

Proof

$$|z_+^i(h) - z_-^i(h)| = \gamma_{+,\delta_i}^i \tilde{z}_{+,\delta_i}(h) - \gamma_{-,\delta_i}^i \tilde{z}_{-,\delta_i}(h) = \quad (2.223)$$

$$= (\gamma_{+,\delta_i}^i - \gamma_{-,\delta_i}^i) \tilde{z}_{+,\delta_i}(h) + \gamma_{-,\delta_i}^i (\tilde{z}_{+,\delta_i} - \tilde{z}_{-,\delta_i})(h) \quad (2.224)$$

$$\geq \inf_{\delta > 0, i \in \mathbb{N}} ((\gamma_{+,\delta}^i - \gamma_{-,\delta}^i) \tilde{z}_{+,\delta}(h)) \geq \quad (2.225)$$

$$\geq \left(\inf_{\delta > 0, i \in \mathbb{N}} \gamma_{+,\delta}^i - \sup_{\delta > 0, i \in \mathbb{N}} \gamma_{-,\delta}^i \right) \tilde{z}_{+,0}(h) \geq 0 \quad (2.226)$$

This proves “ \Rightarrow ”.

Proof of “ \Leftarrow ”: Suppose

$$\inf_{\delta > 0, i \in \mathbb{N}} \gamma_{+,\delta}^i = \sup_{\delta > 0, i \in \mathbb{N}} \gamma_{-,\delta}^i =: \gamma^\infty \quad (2.227)$$

Because of monotonicity in i and δ proved in Lemma 6, we can construct a sequence $\delta_i^* \xrightarrow{i \rightarrow \infty} 0$ such that

$$\gamma_{+,\delta_i^*}^i \searrow \gamma^\infty \quad (2.228)$$

$$\gamma_{-,\delta_i^*}^i \nearrow \gamma^\infty \quad (2.229)$$

Again because of monotonicity we conclude

$$\gamma_{+,\delta_i}^i \searrow \gamma^\infty \quad (2.230)$$

$$\gamma_{-,\delta_i}^i \nearrow \gamma^\infty \quad (2.231)$$

for *every* sequence $\delta_i \xrightarrow{i \rightarrow \infty} 0$, in particular for the one used to construct z_+^i, z_-^i . Therefore

$$\begin{aligned} |z_+^i - z_-^i|_{C^0} &= |\gamma_{+,\delta_i}^i \tilde{z}_{+,\delta_i} - \gamma_{-,\delta_i}^i \tilde{z}_{-,\delta_i}|_{C^0} \leq \\ &\leq |\gamma_{+,\delta_i}^i - \gamma_{-,\delta_i}^i| |\tilde{z}_{-,\delta_i}(h)|_{C^0} + \\ &\quad + |\gamma_{+,\delta_i}^i| |\tilde{z}_{+,\delta_i} - \tilde{z}_{-,\delta_i}|_{C^0} \xrightarrow{i \rightarrow \infty} 0 \end{aligned} \quad (2.232)$$

This concludes step 1.

Step 2

$$\inf_{\delta>0, i \in \mathbb{N}} \gamma_{+, \delta}^i = \inf_{i \in \mathbb{N}} \gamma_{+, 0}^i \quad (2.233)$$

$$\sup_{\delta>0, i \in \mathbb{N}} \gamma_{-, \delta}^i = \sup_{i \in \mathbb{N}} \gamma_{-, 0}^i \quad (2.234)$$

Proof We only prove (2.233), (2.234) is analogous.

Because of monotonicity, “ \geq ” follows at once.

We note that for fixed i ,

$$\gamma_{+, \delta}^i \xrightarrow{\delta \rightarrow 0} \gamma_{+, 0}^i \quad (2.235)$$

Therefore

$$\left| \gamma_{+, \delta}^i - \inf_{k \in \mathbb{N}} \gamma_{+, 0}^k \right| \leq \left| \gamma_{+, \delta}^i - \gamma_{+, 0}^i \right| + \left| \gamma_{+, 0}^i - \inf_{k \in \mathbb{N}} \gamma_{+, 0}^k \right| \quad (2.236)$$

Since $\gamma_{+, 0}^i$ is monotone falling with respect to i ,

$$\gamma_{+, 0}^i \searrow \inf_{k \in \mathbb{N}} \gamma_{+, 0}^k =: \gamma_{+, 0}^\infty \quad (2.237)$$

(likewise, $\gamma_{-, 0}^i \searrow \sup_{k \in \mathbb{N}} \gamma_{-, 0}^k =: \gamma_{-, 0}^\infty$)

Therefore by taking i large enough, we can make the second term in (2.236) smaller than any $\epsilon > 0$; by fixing that i and taking δ small enough, the first term is also made smaller than ϵ .

This means that $\inf_{k \in \mathbb{N}} \gamma_{+, 0}^k =: \gamma_{+, 0}^\infty$ is an accumulation point of $\{\gamma_{+, \delta}^i | i \in \mathbb{N}, \delta > 0\}$, which excludes “ $>$ ” in (2.233).

Step 3

$$\gamma_{+, 0}^\infty = \gamma_{-, 0}^\infty \quad (2.238)$$

if and only if γ_* is the only solution of $G(G(\gamma)) = \gamma$.

Proof The definitions of $\gamma_{+, 0}^i, \gamma_{-, 0}^i$ can be rewritten as

$$\gamma_{+, 0}^i := \frac{\Phi(\gamma_{-, 0}^i \tilde{Z}_{-, 0}(0))}{(a + \chi_{+, 0} d(0))} = \frac{\Phi(\gamma_{-, 0}^i \mathcal{Z}_*)}{(a + \chi d(0))} = G(\gamma_{-, 0}^i) \quad (2.239)$$

$$\gamma_{-, 0}^{i+1} := \frac{\Phi(\gamma_{+, 0}^i \tilde{Z}_{+, 0}(0))}{(a + \chi_{-, 0} d(0))} = \frac{\Phi(\gamma_{+, 0}^i \mathcal{Z}_*)}{(a + \chi d(0))} = G(\gamma_{+, 0}^i) \quad (2.240)$$

since $\tilde{z}_{+,0} = \tilde{z}_{-,0} = z_*$, $\chi_{+,0} = \chi_{-,0} = \chi$.

Therefore $\gamma_{-,0}^i$ are defined by

$$\gamma_{-,0}^1 := 0 \quad (2.241)$$

$$\gamma_{-,0}^{i+1} := G(G(\gamma_{-,0}^i)) \quad (2.242)$$

Let γ_{min} be the smallest solution of $G(G(\gamma)) = \gamma$. Since Φ is positive, $\gamma_{min} > 0$.

Since Φ and therefore G is non-increasing, $G \circ G$ is nondecreasing, in particular

$$\forall \gamma \leq \gamma_{min} \quad G(G(\gamma)) \leq \gamma_{min} \quad (2.243)$$

By induction we conclude $\gamma_{-,0}^\infty \leq \gamma_{min}$. Since G is continuous, $G(G(\gamma_{-,0}^\infty)) = \gamma_{-,0}^\infty$. Therefore we conclude

$$\gamma_{-,0}^\infty = \gamma_{min} \quad (2.244)$$

Now we note that if $\gamma_1 < \gamma_2$ are two different solutions of $G(G(\gamma)) = \gamma$, then $G(\gamma_1) > G(\gamma_2)$.

(Proof: “ \geq ”, as G is non-increasing. If “=” holds,

$$\gamma_1 = G(G(\gamma_1)) = G(G(\gamma_2)) = \gamma_2$$

follows, which is a contradiction.)

Since $\gamma_{+,0}^i = G(\gamma_{-,0}^i)$,

$$\gamma_{+,0}^\infty = G(\gamma_{-,0}^\infty) \quad (2.245)$$

Therefore $\gamma_{+,0}^\infty$ is the *largest* solution of $G(G(\gamma)) = \gamma$. Therefore $\gamma_{+,0}^\infty = \gamma_{-,0}^\infty$ if and only if $G(G(\gamma)) = \gamma$ has only one solution. ■

2.3 Dealing with the variability of shadow

In this section I consider the construction of the coefficient functions (that have been treated as given in all the results above) from the individual's vital rates described in a patch model. If we assume that these vital rates depend only on height and locally available light, then the main objective is describing light variability at a given height from the height structure above it. If this is achieved, then the coefficient functions can be defined as expected values and computed explicitly, as shown below.

2.3.1 The shadow distribution ρ_S

To reconstruct the distribution of light availability at a certain height out of the height structure we make the assumptions

- A: The number of trees per reference surface having height between h and $h + dh$ is Poisson distributed with the expected value $y(h, t)dh$
- B: The amounts of trees per reference surface in different height layers are uncorrelated.

“B” is hardly true in a real forest, but

firstly, comparing the models which use this assumption to empirical data will shed light on the relevance of “B” resp. of the exact spatial structure for the dynamics of the forest,

secondly, in order to reconstruct the probability distribution of light availability from the height structure, *some* such assumption needs to be made,

thirdly, the class of models we arrive at is broad enough to include all PDE-based hierarchical forest models mentioned in section 1.3.2.

We now proceed to derive the formula for the probability density function $\rho_S(S, h, t)$ of the cumulative leaf area above height h which results from our assumptions. (S stands for “shadow”)

The light intensity L is then computed according to the formula

$$L(h, t) = \exp(-C \cdot S) \quad (2.246)$$

where C is a positive constant [25]. Since this is a bijective relationship between L and S , we treat all light-dependent functions as functions of S . Let $\alpha(h)$ denote the amount of foliage per tree of height h . We keep the height h constant, take some big $n \in \mathbb{N}$ and define

$$\Delta h := \frac{h_{max} - h}{n} \quad (2.247)$$

$$h_k := h + k \cdot \Delta h \quad (2.248)$$

Then under assumption “A” the probability density of the number X_k of trees having height between h_{k-1} and h_k is Poisson distributed with mean

$y(h_k, t)\Delta h$ (for n large enough), so $S_k := \alpha(h_k) \cdot X_k$ (the contribution of X_k to S) has the probability density

$$\rho_k(S_k) = e^{-y(h_k)\Delta h} \sum_{m=0}^{\infty} \frac{(y(h_k)\Delta h)^m}{m!} \cdot \delta(S_k - \alpha(h_k) \cdot m) \quad (2.249)$$

Here $\delta(\cdot)$ is the Dirac delta defined by

$$\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0) \quad (2.250)$$

for all continuous f . 2.249 reflects the fact that the number of trees in the height class $[h_{k-1}, h_k[$ is a *discrete* random variable taking only integer values, so S_k can only take the discrete values $m \cdot \alpha(h_k)$, $m \in \mathbb{N}$. Because of assumption ‘‘B’’, the distribution of $S = \sum_{k=1}^n S_k$ is a convolution of all S_k -distributions. Convolution corresponds to multiplication in the Fourier domain, therefore we will do the subsequent computations there.

$\rho_k(S_k)$ has the Fourier transform

$$\begin{aligned} f_k(\xi) &= \exp(-y(h_k)\Delta h) \sum_{m=0}^{\infty} \frac{(y(h_k)\Delta h)^m}{m!} \cdot \exp(-i\xi\alpha(h_k) \cdot m) = \\ &= \exp\left(\left[e^{-i\xi\alpha(h_k)} - 1\right]y(h_k)\Delta h\right) \end{aligned} \quad (2.251)$$

Therefore the probability density of $S(h) = \sum_{k=1}^n \alpha(h_k)X_k$ has the Fourier transform

$$f(\xi) = \prod_{k=1}^n f_k(\xi) = \exp\left(\sum_{k=1}^n \left[e^{-i\xi\alpha(h_k)} - 1\right]y(h_k)\Delta h\right) \quad (2.252)$$

by letting $n \rightarrow \infty$ we get

$$f(\xi) = \exp\left(\int_h^{h_{max}} e^{-i\xi\alpha(h')}y(h', t)dh' - \int_h^{h_{max}} y(h', t)dh'\right) \quad (2.253)$$

$\rho_S(S, h, t)$ is now defined as the inverse Fourier transform of f ,

$$\rho_S(S, h, t) = \check{f}(S) \quad (2.254)$$

Thus we have constructed $\rho_S(S, h, t)$ knowing only the height structure. (the exact formula is probably too clumsy to be used in simulations, but knowing its properties can allow us to construct better approximations).

If the individual growth rate G_{ind} depends only on S, h and some intrinsic (e.g. genetically caused) parameter g having the distribution $\rho_g(g)$, it is now possible to compute

$$M_1^G(h, t) = G(h, t) = \int_0^\infty \int_0^\infty G_{ind}(S, h, g) \rho_S(S, h, t) \rho_g(g) dg dS, \quad (2.255)$$

$$M_n^G(h, t) = \int_0^\infty \int_0^\infty \left(G_{ind}(S, h, g) - G(h, t) \right)^n \rho_S(S, h, t) \rho_g(g) dg dS, \quad (2.256)$$

For comparison, let the individual birth and death probabilities Φ_{ind} and M_{ind} depend deterministically on S and (in case of M_{ind}) on h . Then

$$M(h, t) = \int_0^\infty M_{ind}(S, h) \rho_S(S, h, t) dS \quad (2.257)$$

$$\Phi(t) = \int_0^\infty \Phi_{ind}(S) \rho_S(S, h_0, t) dS \quad (2.258)$$

where $\Phi(t)$ is the total expected number of new saplings (each of which has height h_0). Thus under all these assumptions the height-structured model becomes autonomous, since we can now compute all the coefficients using only information that is included in the model (namely height structure).

2.3.2 Properties of ρ_S

In practice one will probably want to approximate ρ_S by some other function easier to compute. Using basic properties of the Fourier transform, one sees

$$E_{\rho_S}[S] = i (\partial_\xi f) (0) = \int_h^{h_{max}} \alpha(h') y(h', t) dh', \quad (2.259)$$

$$Var_{\rho_S}[S] = - (\partial_\xi^2 f) (0) - E_{\rho_S}^2[S] = \int_h^{h_{max}} (\alpha(h'))^2 y(h', t) dh, \quad (2.260)$$

The total number of trees is equal to

$$n(t) = \int_h^{h_{max}} y(h', t) dh' \quad (2.261)$$

In the only model known to me which is built on these ideas ([3]), ρ_S is approximated by a truncated Normal distribution having this expected value and variance.

But care must be taken in such approximations, since

$$\rho_S(S) = \exp(-n(t)) \cdot \delta(S) + \tilde{\rho}_S(S), \quad \tilde{\rho}_S(S) \geq 0 \quad (2.262)$$

where δ is the Dirac delta. If α is strictly monotone (for instance, $\partial_h \alpha(h) \geq \epsilon > 0 \quad \forall h$) then $\tilde{\rho}_S(S)$ is smooth.

Sketch of a proof: after making in the first integral in the formula 2.253 the variable substitution $\tilde{h} := \alpha(h)$, one sees

$$f(\xi) \xrightarrow{\xi \rightarrow \infty} \exp(-n(t)) \quad (2.263)$$

Therefore

$$\check{f}(S) = \exp(-n(t)) \cdot \delta(S) + \underbrace{(f(\xi) - \exp(-n(t)))^\vee(S)}_{\tilde{\rho}_S(S)} \quad (2.264)$$

Now $f(\xi) - \exp(-n(t)) \xrightarrow{\xi \rightarrow \infty} 0$ as fast as $\int_h^{h_{max}} y(h') \exp(-i\xi \alpha(h')) dh' \xrightarrow{\xi \rightarrow \infty} 0$, therefore $\tilde{\rho}_S(S)$ is smooth.)

This has an immediate consequence for modeling: Let $F_{ind}(S)$ be one of the individual's vital rates, suppose $F_{ind}(0) > 0$ (when full light is available, trees should prosper). Since there is likely to be an a priori bound on $n(t)$ of the form

$$|n(t)| \leq n_{max}, \quad (2.265)$$

we conclude

$$\int_0^\infty F(S) \rho_S(S) dS \geq \exp(-n_{max}) F(0) \quad (2.266)$$

This means that even if the individual vital rates are extremely easily suppressed by shadow (say $F(S) \equiv 0$ for $S \geq S_{crit}$, S_{crit} very small), the corresponding coefficient functions experience at most an exponential decay and are even bounded away from 0 by the positive constant $\exp(-n_{max}) F(0)$, no matter how large $E[S]$ might be. This throws a light on the results of [24] mentioned in section 1.3.2.

2.3.3 An approach for a better approximation of ρ_S

While building a PDE approximation of an individual-based system, one of the most interesting questions is:

What is the simplest approximation of ρ_S still reproducing the original system's dynamics?

[3] hypothesized that using at least $E_{\rho_S}[S]$ and $Var_{\rho_S}[S]$ would be necessary. However, the following idea might allow us to arrive at a decent approximation of ρ_S using $E_{\rho_S}[S]$ only:

We write

$$Var_{\rho_S}[S](h) = \int_h^{h_{max}} (\alpha(r))^2 y(r) dr = E_{\rho_S}[S](h) \int_h^{h_{max}} \underbrace{\frac{\alpha(r)y(r)}{E_{\rho_S}[S](h)}}_{\rho_{\#}(r)} \alpha(r) dr \quad (2.267)$$

now since $\int_h^{h_{max}} \rho_{\#}(r) dr = 1$, and α is continuous, we conclude

$$Var_{\rho_S}[S](h) = E_{\rho_S}[S](h) \cdot \alpha(\tilde{h}) \quad \text{for some unknown } \tilde{h} \in [h, h_{max}] \quad (2.268)$$

Now \tilde{h} is, of course, dependent on ρ_S and therefore on $y(\cdot, t)$; however, one could try approximating \tilde{h} (however crudely) by a function of h only. Then we would get an approximation of $Var_{\rho_S}[S](h)$ through a function of h and $E_{\rho_S}[S](h)$ only, whereupon we could proceed to constructing an approximation of ρ_S by the method of [3] or some other, similar method which would also take into account our remarks on the form of ρ_S made in section 2.3.2. Then the expected values that we use as coefficient functions would depend on $E_{\rho_S}[S](h)$ and h only, as assumed in the sections 2.1 and 2.2.

If this approximation proves to be adequate, it would firstly provide a general way of constructing the coefficient functions that have been considered as given in the previous sections and secondly, lead to a simpler model than that of [3] that would be just as good an approximation of the corresponding patch model.

Chapter 3

Conclusion

3.1 Summary

In this thesis I have considered a general second-order parabolic PDE representing a one-species height-structured forest model. The coefficients of the equation have been allowed to depend on height h and some weighted integral of the tree density $y(h, t)$ above h . (This integral can be thought of, for instance, as the expected cumulative leaf area above h).

The restrictions on the coefficient functions dealt only with regularity and pointwise bounds, without any more specific assumptions. For such a system existence and uniqueness of solutions has been proved under the assumption of a pointwise a priori bound (depending on the initial condition) on the tree density:

$$\sup_{h,t} |y(h, t)| \leq \mathcal{C}(y(\cdot, 0)) \tag{3.1}$$

Since such a bound is a property of the system to be expected from the biological point of view, it should be easily proved for any particular system of this kind, making this result a useful tool for study of all such systems.

A special case was presented, with a certain form of the nonlinear expected growth rate and with a linear variance of the growth rate (diffusion constant) fulfilling some technical conditions. (These conditions were weak enough to include, for instance, the models with an arbitrarily small (nonzero) diffusion coefficient as well as a wide class of diffusion coefficients having the same order of magnitude as the experienced growth rate.) In this case, the problem

could be transformed into a form allowing liberal application of the maximum principle.

By using that, I could construct a series of ever improving lower and upper asymptotic bounds for any solution of the problem.

The question, whether these lower and upper bounds converge against the same limit, which would imply a unique globally attractive steady state independent of the initial conditions, as observed in the simulations by [1], [2] and [3], was examined on a simplistic special case and had to be answered in the negative, unless additional assumptions on the coefficient functions were made.

Further I considered the question of constructing the coefficient functions that in our equation describe birth, mortality and growth, from the corresponding functions describing an individual in the patch model.

As the work of [24] and [3] discussed in the introduction shows, this is not a trivial question. The only successful attempt so far is found in [3], who sets the coefficient functions equal to expected values of the individual vital rates respective to the light distribution of a given height level (or rather some approximation of this distribution). I built on this idea, firstly obtaining a better description of the light distribution (given the same assumptions as in [3]), and secondly proposing a method to approximate it using only one weighted integral of the tree density instead of two, as in [3].

One implication relevant for modeling is that under the assumptions of [3] there is always at any height a finite fraction of trees that is not overshadowed at all. Therefore, the coefficient functions in the PDE model should be chosen strictly positive for any values of the expected cumulative leaf area, even if individual growth takes place only given sufficient light.

Unfortunately, because of the strict time limit of a diploma thesis, I am not yet able to provide many results with direct relevance to ecology. However, in the next section numerous directions for further work are discussed that should lead to such results.

I conclude that the class of models considered in this thesis combines significant realism with analytical tractability, so its further investigation is worthwhile.

3.2 Outlook

Here I briefly sketch some open questions arising from the results proved and propose some ways of addressing them

3.2.1 Further study of PDE systems.

The existence and uniqueness results proven in this thesis should immediately generalize to many-species systems interacting via light only as well as to coefficient functions depending on several differently weighed integrals of the tree density, as long as the dependence is smooth.

An important variation of our system is the case where both the expected value and the variance of the growth rate go to zero at the right boundary, as in [1]. The problem is then longer parabolic, but could be transformed into such by an appropriate variable transform $x = x(h)$. For the diffusion constant of the new system to be bounded away from 0 by a positive constant,

$$x(h) \rightarrow \infty \quad \text{for} \quad h \searrow h_{max} \quad (3.2)$$

is necessary. Thus we could reduce this non-parabolic problem with a finite right boundary to a problem of the kind considered in this thesis, but with arbitrary large x allowed. Thus the problem formulation with open right boundary acquires immediate importance.

One could ask following questions concerning the coefficient functions:

a) The birth function Φ .

What shape should Φ have?

The birth functions considered throughout this thesis were strictly decreasing with respect to the amount of parent trees. This modelled the suppression by overshadowing but ignored the fact that seed supply depends on parent tree abundance. In a patch model, such a simplification is almost inevitable since seed transport is a nonlocal process, the most “natural” way of modelling it in a local manner is assuming a constant seed supply. In hight-structured models of whole forests, however, one should be able to work with more realistic Φ tending to zero if parent tree abundance tends to zero. The problem is that in the simulations of [3] introducing such a dependence on parent tree abundance led to extinction of many species that were present in

the “normal” run as well as in the ForClim patch model ([25]) and in reality. An analytical understanding of this phenomenon is still lacking.

Can some Φ give rise to instability?

An open question is the qualitative behaviour of even the one-species solution in the case where the lower and upper bounds constructed in this thesis do not converge against the same limit: Can then instability or multiple stable equilibria appear - or could one with a more refined approach still prove the existence of a globally attractive equilibrium?

I would conjecture the former. Consider Φ being a step function positive for small Y and suddenly becoming zero shortly before the Y -value corresponding to an equilibrium is reached (or a slightly smoothed version of this step function). In this case it seems probable oscillations would occur.

b) The growth expectation value G .

Simulations suggest that the nonlinearity in w_L greatly reduces the effect on the steady-state distribution of changing the value of the birth function (see Section 1.3.2). One could try to demonstrate this at least by linearisation around the steady state.

c) The growth variance D and the mortality m .

Are there any special effects generated by the nonlinearities in D and m as compared to the linear case?

What character will the dependency on Y will $D(Y, h)$ have for fixed h ? (as compared to the monotonicity properties of m and G)

To answer the questions above, one might take the following approaches:

A first step towards understanding the effects of different kinds of birth functions could be investigation of the simplistic system described in Section 2.2.1. There it shouldn't be hard to understand what kind of dynamics a particular shape of Φ leads to.

A corresponding toy problem to improve the understanding of *systems* would be a system of linear equations with nonlinear boundary conditions. There the question of an attractive equilibrium should reduce to a scalar iteration of $\gamma_{+/-,s}^i$. In fact if the birth functions all depend only on the *total* Y , then it should reduce to iterating *one* scalar variable by taking an appropriate sum.

Effects of the nonlinearities in G , D and m could be investigated in a similar manner by making all the coefficient functions linear except the one whose influence we like to study.

Stability of an equilibrium in the case where the bounds constructed in this thesis don't converge towards the same limit could be investigated via linearisation around the steady-state solution.

Answering these questions should pave the way towards understanding the problems that concern the ecologists, namely:

Given a multi-species model with concrete coefficient functions,

- How many equilibria exist?
- Are they stable?
- How many species can coexist in an equilibrium?

3.2.2 Building PDE systems

An important direction of further studies concerns the construction of concrete coefficient functions for our model given a patch model to be approximated.

The central question here is constructing approximations of the light distribution on any given height that use as little information as possible (for improved tractability and simulation time) while still leading to a satisfactory approximation of the patch model's dynamics.

Connected with this is the question on what variables the coefficient functions should depend on, and what character that dependence should have.

3.2.3 Numerical experiments

The numerical simulation could be used to check whether the coefficient functions constructed according to the general recipes do indeed reproduce the dynamic of the corresponding patch model sufficiently well.

Even if we have no sharp criteria for the existence of a globally attractive steady state one could still compute the lower and upper bounds of Theorem 3 numerically for any particular system, and their numerical convergence towards the same limit (if it occurs) would then still be a proof for the existence of a globally attractive stable state in that particular case.

An interesting object for numeric experiments would be the case where the asymptotic lower and upper limits do *not* converge against each other. Simulation of the solution's behaviour in this case would point out the analytic results we should be looking for: Whether there are several steady states, instability or still a globally attractive steady state whose existence can be proven by some other method than the one employed in the present thesis.

Appendix A

Some interpolation inequalities

Lemma 7 $\forall f(x) \in C^1([0, \infty[)$

$$\left(\langle f \rangle^{(\frac{1}{2})} \right)^2 \leq 2 |g|_{C^0} |g'|_{C^0} \quad (\text{A.1})$$

Proof: Since $x \mapsto x^2$ is continuous and strictly increasing on \mathbb{R}_+ ,

$$\left(\sup_{0 < |x-y| \leq \rho} \frac{|f(x) - f(y)|}{|x-y|^{\frac{1}{2}}} \right)^2 = \sup_{0 < |x-y| \leq \rho} \frac{|f(x) - f(y)|^2}{|x-y|} \leq 2 |g|_{C^0} |g'|_{C^0}$$

■

Lemma 8 $\forall f(x) \in C^2([0, \infty[)$

$$|f'|_{C^0}^2 \leq 4 |f|_{C^0} |f''|_{C^0} \quad (\text{A.2})$$

Proof: it is enough to prove that $|f'(0)|^2 \leq 4 |f|_{C^0} |f''|_{C^0}$, since we can then apply that on $f|_{[x, \infty[}$ for all x to prove (A.2)

Since the equation to be proved is invariant under scaling (both in f and x), we can without loss of generality assume

$$|f|_{C^0} = 1, \quad |f''|_{C^0} = 1$$

Now

$$\forall x \in \mathbb{R}_+ \quad f(x) = f(0) + f'(0) \cdot x + f''(\xi) \cdot \frac{x^2}{2}$$

we set $x = |f'(0)|$ and obtain

$$2 \geq |f(x) - f(0)| = \left| f'(0) \cdot x + f''(\xi) \cdot \frac{x^2}{2} \right| \geq \frac{1}{2} |f'(0)|^2$$

which finishes the proof. ■

Now let $0 < \alpha < 1$. We define

$$\langle f \rangle^{[\alpha]} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

(as opposed to $\langle f \rangle^{(\alpha)} := \sup_{0 < |x-y| \leq \rho} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$ as defined in [27] and used in this thesis)

Lemma 9 $\exists C \in \mathbb{R} : \quad \forall f(x) \in H^{(2+\alpha)}([0, \infty[)$

$$|f''|_{C^0} \leq C \left(|f|_{C^0} \right)^{\frac{\alpha}{2+\alpha}} \cdot \left(|f|^{(2+\alpha)} \right)^{\frac{2}{2+\alpha}} \quad (\text{A.3})$$

Proof: we start by proving

$$|f'|_{C^0} \leq 8 \left(|f|_{C^0} \right)^{\frac{\alpha}{1+\alpha}} \cdot \left(\langle f' \rangle^{[\alpha]} \right)^{\frac{1}{1+\alpha}}. \quad (\text{A.4})$$

As in the previous lemma it suffices to estimate $|f'(0)|$ under the assumption

$$|f|_{C^0} = 1, \quad \langle f' \rangle^{[\alpha]} = \frac{1}{2}$$

Now

$$\begin{aligned} |f'(0) \cdot x| &\leq |f(x) - f(0) - f'(0) \cdot x| + |f(x) - f(0)| \leq \\ &\leq 2 + |(f'(\xi) - f'(0)) \cdot x| \leq 2 + \frac{1}{2} |x|^{1+\alpha} \end{aligned}$$

We set $x = |f'(0)|^{1/\alpha}$ and it follows

$$\frac{1}{2} |f'(0)|^{\frac{1+\alpha}{\alpha}} \leq 2 \iff |f'(0)| \leq 4^{\frac{\alpha}{1+\alpha}} \leq 4$$

which proves (A.4). We now apply (A.4) on f' and use Lemma 8 to achieve

$$|f''|_{C^0} \leq 8 \left(|f'|_{C^0} \right)^{\frac{\alpha}{1+\alpha}} \cdot \left(\langle f'' \rangle^{[\alpha]} \right)^{\frac{1}{1+\alpha}} \leq 16 \left(|f|_{C^0} |f''|_{C^0} \right)^{\frac{\alpha}{2(1+\alpha)}} \cdot \left(\langle f'' \rangle^{[\alpha]} \right)^{\frac{1}{1+\alpha}}$$

which is equivalent to

$$|f''|_{C^0} \leq C \left(|f|_{C^0} \right)^{\frac{\alpha}{2+\alpha}} \cdot \left(\langle f'' \rangle^{[\alpha]} \right)^{\frac{2}{2+\alpha}}$$

Now since $\langle f'' \rangle^{[\alpha]} \leq C |f|^{(2+\alpha)}$, the proof is complete

■

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